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APPLICATION OF FUNCTIONAL ANALYSIS OPTIMIZATION TECHNIQUES TO THE ECONOMIC SCHEDULING OF HYDRO-THERMAL ELECTRIC POWER SYSTEMS

by

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A THESIS

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ABSTRACT

In this thesis problems of optimum scheduling of hydro-thermal power systems are discussed. The scheduling problems are solved by the use of functional analysis, and in this case, the minimum norm formulation is employed.

Optimal schedules are derived for the classical all-thermal power system problem. Here the scheduling equations are shown to be equivalent to earlier results obtained using other optimization techniques. The problem of hydro-thermal systems scheduling when the hydro plants are located on separate streams is treated. One, a system with fixed head hydro-plants and negligible transmission loss is considered. Two, a fixed head hydro plants system where transmission losses are included is presented. Three, a power system is considered where head variations at the hydro-plants are not negligible. In each case the optimal schedule is obtained. The problem of actually implementing the optimal schedules is discussed. The scheduling equations obtained are shown to be equivalent to those obtained using other techniques.

New scheduling equations are developed for the common-flow hydro-plants of the system. Here the time taken by water to flow from the upstream plant to the downstream plant is taken into account. Also, the tail-race elevation at the hydro-plants is considered.

In the above problems the general loss formula is employed to represent the electric network of the system. However, in the final

chapter of this thesis, the problem of the optimal hydro-thermal load flow is investigated. Here the exact model of the electric network is employed. Reliability considerations are also incorporated in the formulation. Finally, it is shown in the last chapter how the efficiency variations and trapezoidal reservoirs effects can be included in the formulation.

The computational aspects of the obtained scheduling equations are discussed. Practical examples are also given to illustrate the results obtained.

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LIST OF SYMBOLS

$A_{m+i}(t)$	A known function of time corresponding to the data of the hydro-plant.
$\underline{B}(t)$	A square symmetric positive definite matrix.
B_{m+i}	A known constant corresponding to the data of the hydro-plant.
B_{io}	Linear coefficient in the loss formula.
B_{ij}	Quadratic coefficient of the loss formula.
B^{ij}	Shunt susceptance between the i<u>th</u> and j<u>th</u> modes in mhos.
b_i	A given constant corresponding to the volume of water discharged over the optimization interval.
C_i	A constant.
C_{m+i}	A constant corresponding to the hydro-plants' data.
$D_i(t)$	Volume of water in the reservoir if there is no discharge.
E_i	A constant corresponding to the hydro-plants' data.
$E_i(t)$	Voltage magnitude at the i<u>th</u> bus.
$E_{d_i}(t)$	Direct axis voltage at the i<u>th</u> bus.
$E_{q_i}(t)$	Quadrature axis voltage at the i<u>th</u> bus.
F_i	Fuel cost function of the i<u>th</u> thermal plant.
$G_i(t)$	Hydro inverse efficiency function.
G^{ij}	Shunt conductance between the i<u>th</u> and j<u>th</u> nodes.
$h_i(t)$	Effective hydraulic head.
$\underline{I}$	The identity matrix.
$I_i(t)$	Volume of water inflow to the i<u>th</u> reservoir.

$i_i(t)$	Rate of water inflow to the ith reservoir.
$J(.)$	A cost functional.
K_i	A given constant
$k_i(t)$	Unknown function.
K_{ij}	Reliability coefficient.
$\underline{L}(t)$	Auxiliary vector function.
$\underline{M}(t)$	Auxiliary vector function.
m	Number of thermal plants in the system.
N	Number of nodes in the system.
N_g	Number of generator buses in the system.
N_h	Number of hydro generating buses.
$N_i(t)$	A known function of time.
$n_i(t)$	An unknown function of time.
$\underline{0}$	The zero matrix.
$P_D(t)$	Active power demand.
$P_L(t)$	Active power loss.
$P_i(t)$	Active power into the ith bus.
$P_{s_i}(t)$	The ith thermal plant active generation.
$P_{h_i}(t)$	The ith hydro-plant active generation.
$Q_i(t)$	Volume of water discharged at the ith plant. (In chapter 6, this denotes the reactive power into the ith bus).
$Q_{W_i}(t)$	Volume of water discharged at the ith plant.
$q_i(t)$	Rate of water discharge at the ith plant.
$\underline{R}(t)$	An auxiliary vector function.
R^n	An n-dimensional real space.

$r_i(t)$	An unknown function of time.
S_i	The reservoir's surface area.
$S_i(t)$	Volume of water stored at the ith reservoir.
T	A bounded linear transformation.
T^*	Conjugate of T.
$T^\dagger$	Pseudo-inverse of T.
T_f	Final time.
$\underline{U}(t)$	Control vector function.
$\underline{V}(t)$	Auxiliary vector function.
$\underline{W}(t)$	Control sub-vector corresponding to the hydro-plants' dynamics.
$x_{m+i}(t)$	A pseudo-control variable.
$y_i(t)$	Forebay elevation at the ith hydro-plant.
$y_{T_i}(t)$	Tail-race elevation.
α_i	Thermal fuel cost coefficient.
α_{m+i}	A hydro-plant's constants.
β_i	Thermal fuel cost coefficient.
β_{y_i}	Inverse forebay area.
β_{T_i}	Tail race coefficient.
γ_i	Thermal fuel cost coefficient.
δ_{g_i}	Efficiency function coefficients.
$\underline{\xi}$	A given vector.
η_i	Efficiency of the ith hydro-plant.
$\theta(t)$	Unknown function of time.

$\lambda(t)$	Unknown function of time.
col.[.]	A column vector of the arguments.
diag[.]	A diagonal matrix with elements as the arguments.

CHAPTER I

INTRODUCTION

1.1 Background

A prime objective in the operation of a power system is to achieve optimum economic dispatch. This is the problem of scheduling the generation at various generating plants. Here the system's power demand is to be supplied at the lowest possible power production cost. Normally this is planned for an appropriate time interval. The optimum operation of a power system will also depend upon restrictions imposed by factors other than operating economics. Possible decreases in power production costs of only fractions of a per cent are still of vital concern to the electric utility industry. In addition, the capability to solve the economic dispatch problem is extremely useful for the planning and design of future equipment additions to power systems. For these reasons, the economic dispatch problem has been the subject of extensive research [1,2].

In economic dispatch it is customary to consider the operating costs only. This ignores expenses of capital, labour, start-up and shut-down related to the length of the outage period for a certain unit. It is essential to have an accurate knowledge of the manner in which the total cost of operation of each available energy source varies with the instantaneous output. In most cases, the fuel supply available to the conventional thermal plant is not a limiting factor in the operation of the plant. In such cases the appropriate price to use for economic dispatch is the

current cost of incoming fuel adjusted for handling costs, maintainance cost of fuel handling facilities, etc. In this thesis it is assumed that the fuel cost is a second order polynomial of the power output [3].

The hydro-thermal optimization problem is different from the all-thermal one. The former involves the planning of the usage of a limited resource over a period of time. The resource is the water available for hydro-generation. Most of the hydro-electric plants are multipurpose in nature. In such cases it is necessary to meet certain obligations other than power generation. These may include a maximum forebay elevation not to be exceeded due to flood prospects and a minimum plant discharge and spillage to meet irrigational and navigational commitments. Thus the optimum operation of the hydro-thermal system depends upon the conditions which exist over the entire optimization interval [4]. Many systems with large water storage capacity will require a year for the optimization interval. Another system may have run-of-the-river plants with only a small or moderate storage capacity. An optimization interval of a day or a week may be useful in this case [5].

Other distinctions among power systems are the number of hydro stations, their location and special operating characteristics. The problem is quite different if the hydro stations are located on the same stream or on different ones. In the former case, the water transport delay may be of great importance [6, 7, 8]. An upstream station will highly influence the operation of the next downstream station. The latter, however, also influences the upstream plant by its effect on the tail water elevation and effective head. Close coupling of stations by such a phenomenon is a complicating factor.

1.2 Scope of the Thesis

In this thesis, optimum generation schedules will be developed for a hydro-thermal electric power system. The scheduling problem is solved by use of functional analysis where the minimum norm formulation is employed. The solution found here is guaranteed to be the unique optimal solution [9]. The power system considered contains an arbitrary number of plants.

Aspects of functional analysis optimization techniques that will be applied to the power system problems are presented in Chapter II. In this presentation it is chosen to start with the simplest possible problem and then gradually lead up to the more complex problems.

In Chapter III, we consider an all-thermal electric power system with negligible transmission losses. This problem is a classical one [10]. It is included here to show the application of the minimum norm formulation in the case of additive linear transformations.

Chapter IV is concerned with hydro-thermal system with hydro plants on separate streams. The electric network representation relies on the active power balance equation. Section 4.1 covers the background and previous work done in this area using other optimization techniques. Attention is focused in this chapter on the hydraulic head variations in the case of vertical sided reservoirs and transmission losses. In Section 4.2 a system with fixed head hydro plants and negligible transmission losses is considered. The assumption of negligible transmission losses is relaxed in Section 4.3. Finally, the more general problem of variable head hydro plants with transmission losses in the active power balance equation is presented in Section 4.4. The results

obtained are shown to agree with Kron's equations [11]. The practical implementation of actually finding the optimum generation schedules is illustrated by way of two examples in sub-sections 4.3.5 and 4.4.6. Here functional analytic computational search techniques are employed.

In Chapter V the problems of common-flow between plants are considered. Here the time delay of flow between hydro plants on the same stream is taken into account. The effect of tail water elevation on the effective head at the hydro plants is considered. The problem of the power system with variable head hydro plants on the same stream is the subject of Section 5.2. Here a practical example showing the computational aspect of the problem is given. The computational scheme employed here is again a functional analytic one. A general power system in as far as the relative locations of the hydro plants are concerned is treated in Section 5.3.

A more complex problem in the area of economic scheduling of power systems is that of optimal load flows. This is the subject of Chapter VI. Here a more realistic representation of the electric network is adopted. Starting from the basic model, a suitable form of the load flow equation is derived to facilitate the mathematical formulation. All of the electric variables of the system are considered in this chapter. Here we also include the more ambitious objective of system reliability. Also inequality constraints imposed on the network variables are considered. Furthermore, consideration is given to variable efficiency and trapezoidal reservoirs hydro plants at the end of this chapter.

Only the titles and the broad outline of the problems considered in this thesis are mentioned here. A more detailed description of each

of the problems and its relationship to the previous work in this area, will be found at the beginning of each chapter.

CHAPTER II

THE FUNCTIONAL ANALYTIC OPTIMIZATION TECHNIQUE

2.1 Background

During the years optimal control theory was developed, powerful general solution methods were introduced. These are based on the now widely known "Maximum Principle" and "Optimality Principle". Parallel to the development of these, starting in 1956, attempts have been made to introduce methods of functional analysis into the study of optimal control problems.

At first it seemed that the methods of functional analysis applied only to a very restricted class of problems. But in spite of this, the number of studies using the ideas of functional analysis has increased. In solving optimal control problems, by using the Maximum Principle, or by reduction to the Euler equations, these methods do not show how to select the initial conditions required for solving the adjoint system. The methods of Dynamic programming and the approach that leads to the Hamilton-Jacobi equations do not have this deficiency. However, the solution of functional equations is not an easy problem [13].

One of the typical features of the functional analysis approach is that it yields necessary and sufficient conditions for the existence of solutions. This fact makes it possible to study the qualitative aspects of optimal processes. Moreover, this approach is free of the concrete nature of the system. Thus many formulations hold for systems that are distributive, digital, composite, nonlinear or biological. Of course, results obtained on the basis of an abstract formulation must

then be given concrete identification in its various physical forms.

Below, we give a survey of certain works where the methods of functional analysis are used in solving problems in the theory of optimal processes. It is not the author's intention to give a complete exposition of the application of functional analysis to the theory of optimal processes. In fact, works dealing with the abstract minimum norm formulation will be our main concern.

Investigation of the problem of approximate solutions to first order ordinary differential equations, led D.S. Carter [14], in 1957, to the earliest minimum norm formulation. Carter's problem was concerned with obtaining an element of a specific Banach space. The image of the element sought under a first order linear differential operator was to be of minimum norm. The norm adopted was the maximum norm. The element was to satisfy a two-point-boundary condition.

In 1962, W.T. Reid [15] extended Carter's results to the case of an nth order differential operator. This was achieved by reducing the problem to a problem in the theory of moments. The general results of the Hahn-Banach theorem were then applied to the reduced problem.

A minimum norm problem in Hilbert spaces was considered by A.V. Balakrishnan [16] in 1962: Given a compact, bounded, linear operator mapping H_1 into H_2 and with H_1 and H_2 being Hilbert spaces an element of a given ball in H_1 is sought. The norm of the difference between the image of this element and a given element in H_2 is to be a minimum. A sequence of elements in H_2 was obtained and was shown to converge to the desired element.

A wide class of minimum norm problems was considered in 1962 by L.W. Neustadt. In his paper [17], Neustadt employed a variational

approach to find the minimum norm element. This was set in the Banach spaces of the L_p type ($1 \leq p \leq \infty$). The system satisfied a linear integral operator and the problem was reduced to minimizing a functional of a new variable. This variable is in many ways analogous to the co-states of the Pontryagin's Maximum Principle. The method of steepest-descent was suggested for implementing the final optimal.

G.M. Kranc and P.E. Sarachick [18] considered a minimum norm problem in 1963. This was essentially the same as Neustadt's problem. The only exception was that the element (control) sought was to belong to a specified ball in the L_p space under consideration. Hölder's inequality was used extensively to specify the optimal solution.

While most authors chose to consider relatively well specified systems, W.A. Porter chose a more abstract approach to the optimization problem. Knowing that diverse systems of equations can be associated with linear transformations, Porter (1964) considered a problem involving a linear transformation on a "Hilbert space". The cost associated with an element of the Hilbert space was given by the Hilbert space norm [19]. Later together with J.P. Williams [20, 21] he extended the results of this abstract problem to cases involving Banach spaces.

It was noted that the results of these approaches are applicable to systems of discrete, continuous and composite types. These results can be utilized for various optimization problems [22, 23, 24]. The work of this dissertation involves a recognition of some of these areas.

2.2 Some Functional Analysis Concepts

The discussion of this section is aimed at displaying some of the

concepts and symbols which are utilized in the next section. Basic functional analysis concepts are left to the references of which [9, 25, 26, 27, 28] are well suited.

The next section will be concerned with linear spaces which have norms defined upon them. A norm (commonly denoted by $||.||$) is a real valued, positive definite ($||x|| > 0$ for $x \neq 0$), absolutely homogeneous ($||\lambda x|| = |\lambda|.||x||$), and subadditive ($||x + y|| \leq ||x||+||y||$) function.

A transformation is a mapping from one vector space to another. If T maps the space X into Y, we write $T: X \rightarrow Y$. If T maps the vector $x \varepsilon X$ into the vector $y \varepsilon Y$, we write $y = T(x)$ and y is referred to as the image of x under T. Alternatively a transformation is referred to as an operator.

The transformation $T: X \rightarrow Y$ is said to be linear if for every x_1, $x_2 \varepsilon X$ and all scalars α_1 and α_2 one has $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$. The linear operator T from a normed space X to a normed space Y is said to be bounded if there is a constant M such that $||Tx|| \leq M||x||$ for all $x \varepsilon X$. The normed space of all bounded linear operators from the normed space X into the normed space Y is denoted by $B(X,Y)$.

A functional is a transformation from a linear space into the space of real (or complex) scalars. A functional f on a linear space X is linear if for any two vectors $x, y \varepsilon X$ and any two scalars α and β there holds $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$. A linear functional f on a normed space is bounded if there is a constant M such that $|f(x)| \leq M||x||$ for every $x \varepsilon X$. The smallest such constant M_0 is called the norm of f. The norm of the functional f can be expressed as

$$||f|| = \sup_{x \neq 0} |f(x)|/||x|| \tag{2.1}$$

Given a normed linear space X, one can define bounded linear functionals on X. The space of these linear functionals is a normed linear space X^*. The space X^* is the normed dual of X (alternatively X^* is called the conjugate space of X), and is a Banach space. If X is a Hilbert space, then $X = X^*$. Thus Hilbert spaces are self-dual. The normed dual X^{**} of X^* is called the second dual space of X. A normed linear space X is said to be reflexive if $X = X^{**}$. Any Hilbert space is reflexive.

Let X and Y be normed spaces and let $T \varepsilon B(X,Y)$. The adjoint (conjugate) operator T^*: $Y^* \to X^*$ is defined by

$$\langle x, T^* y \rangle = \langle Tx, y^* \rangle$$

An important special case is that of a linear operator T: H→G where H and G are Hilbert spaces. If G and H are real then they are their own duals and the operator T^* can be regarded as mapping G into H. In this case the adjoint relation becomes

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

Let G and H be Hilbert spaces and $T \varepsilon B(G,H)$ with the range of T being closed. Define the set M as $M = \{x_1 \varepsilon G: ||Tx_1 - y|| = \min_x ||Tx - y||\}$.

Let $x_o \varepsilon M$ be the unique vector of minimum norm. Then the pseudo-inverse operator $T^\dagger$ of T is the operator mapping y into its corresponding x_o as y varies over H.

The Minkowski functional p of a convex set K in a normed linear space X is defined on X by:

$$p(x) = \inf\{r: \frac{x}{r} \varepsilon K, r>0\} \tag{2.2}$$

In a Banach space B, the set $U = \{x \varepsilon B: ||x|| \leq 1\}$ is said to be the Closed Unit Ball in B. The boundary ∂U of U is called the unit sphere $\partial U = \{x \varepsilon B: ||x|| = 1\}$. The Banach space B is called rotund (strictly convex) if ∂U contains no line segments. Any convex set K in a rotund space has at most one minimum element. The Banach space B is said to be smooth if at each point of ∂U there is exactly one supporting hyper-plane of U.

A vector $x \varepsilon U$ is called an extremal of $f \varepsilon B^*$ if x satisfies $f(x) = ||f||$. At most one extremal of f exists if and only if B is rotund. If $x \varepsilon B$ then $f \varepsilon U^*$ is an extremal of x if f satisfies $f(x) = ||x||$. At most one extremal of x exists if and only if B is smooth. If $x \varepsilon B$, the Hahn-Banach theorem [9,28] guarantees the existence of at least one extremal f in B^*. Accordingly, if B is reflexive, every $f \neq 0$ and $f \varepsilon B^*$ has a unique extremal in B if and only if B is rotund. Every $x \neq 0$ and $x \varepsilon B$ has a unique extremal in B^* if and only if B^* is smooth. Finally, if B is reflexive, rotund and smooth and we denote by $\overline{x}$ the unique extremal of x in B^*, then $\overset{=}{x} = \frac{x}{||x||}$ for all $x \neq 0$ and $x \varepsilon B$.

2.3 The Minimum Norm Problem

The main results of this dissertation are based on previous analysis by Porter and Williams [19, 20, 21] of an abstract minimum norm problem. For background purposes this section considers this problem which may be formulated as follows:

> *Let B and D be Banach spaces. Let T be a bounded linear transformation defined on B with values in D. For each ξ in the range of T, find an element $\mu \varepsilon B$ that satisfies*
>
> $$\xi = T\mu$$
>
> *while minimizing the performance index*
>
> $$J(\mu) = ||u||$$

The solution of this problem as obtained by Porter and Williams will be given in the form of theorems.

Theorem 1 :(Existence)

The minimum norm problem formulated above has a solution for every bounded linear transformation defined on B if and only if B is reflexive.

Theorem 2 :(Uniqueness)

For every $\xi \varepsilon D$, a unique optimal solution exists if B is reflexive, rotund and smooth.

Theorem 3 :(Characterization)

With the conditions of theorems 1 and 2 satisfied there exists

a unique unit $\phi_\xi \; \varepsilon D^*$ for every $\xi \varepsilon D$ which defines the optimal u denoted by u_ξ as:

$$u_\xi = T^\dagger \xi = p(\xi)\overline{T^* \phi_\xi} \tag{2.3}$$

ϕ_ξ is determined by the conditions

1. $||\phi_\xi|| = 1$
2. Either:

   a. $<\eta,\phi_\xi> \leq [p(\xi)]^{-1}<\xi,\phi_\xi>$ for all $\eta\varepsilon C$

or

   b. $||T^*\phi_\xi|| = [p(\xi)]^{-1}<\xi,\phi_\xi>$

Theorem 3 defines the optimal solution in an implicit form. To eliminate the ambiguity the following transformations are defined:

1. Define the transformation K: $B^* \to B$ by the relation

$$K(f) = ||f||\overline{f} \quad \text{for every } f\varepsilon B^* \tag{2.4}$$

Thus K is an extremizing norm restoring operator.

2. Define the transformation J: $D^* \to D$ by the relation

$$J(\phi) = ||T^*\phi||T(\overline{T^*\phi}) \quad \text{for every } \phi\varepsilon D^* \tag{2.5}$$

Thus

$$J(\phi) = T(K(T^*\phi)) \tag{2.6}$$

or

$$J = TKT^* \tag{2.7}$$

Thus theorem 3 can be restated as

<u>Theorem 3 :(Modified Optimal Characterization)</u>

The unique optimal $u_\xi \varepsilon B$ is given by:

$$u_\xi = T^\dagger \xi \tag{2.8}$$

where the pseudo-inverse operator $T^\dagger$ is given by

$$T^{\dagger}\xi = KT^{*}J^{-1}\xi \tag{2.9}$$

In Hilbert spaces the operator K reduces to the identity operator so that

$$T^{\dagger}\xi = T^{*}[TT^{*}]^{-1}\xi \tag{2.10}$$

provided that the inverse of TT^{*} exists.

An extension of the results of the minimum norm problem discussed in this section is as follows:

Let B, D, T and ξ be as in the minimum norm problem. Let $\hat{u}$ be a given vector in B. Then the unique $u_{\xi} \varepsilon B$ satisfying

$$\xi = Tu$$

which minimizes the performance index:

$$J(u) = ||u - \hat{u}||$$

is given by

$$u_{\xi} = T^{\dagger}[\xi - T\hat{u}] + \hat{u} \tag{2.11}$$

CHAPTER III

A THERMAL POWER SYSTEM

3.1 Background

This chapter is devoted to the problem of optimizing the operation of an exclusively thermal electric power system. Much of the early investigations in the economy loading field were centred upon steam-electric generating units operating in parallel. This resulted in the development of the theory of incremental loading [10, 29, 30]. The method leading to this theory successfully employed simple calculus techniques.

Achieving minimum total cost of supplying the power requirements of a system requires an accurate knowledge of the operating cost functions. A cost function represents the manner in which the total cost of operation of a generating unit varies with its output. The total cost of operation includes the fuel cost, cost of labour, supplies and maintenance. However, no methods are presently available for expressing the latter as a function of the output [1, 10, 31]. Arbitrary methods of determining these costs are used. The most common one is to assume the cost of labour, supplies and maintenance to be a fixed percentage of the incoming fuel costs. It is common practice to obtain the operating cost function by establishing the input-output curve for the plant considered, then adjusting for the cost of the fuel per unit input to the plant. Throughout this dissertation it is assumed that the cost

functions may be approximated by second order polynomials of the instantaneous output power of the thermal plants [3, 32].

The problem considered in this chapter deals with the case when all sources of generation are either located at the same bus or close enough so that transmission losses may be neglected. In this case the instantaneous system power demand is equal to the sum of all available instantaneous generations. Accurate forecasting of the hourly power demand is thus an important factor in any dispatch strategy. This depends on both weather information and historical patterns. Procedures for active power demand forecasting are in existence [33]. In this thesis it is assumed that the daily power demand of the system is known beforehand.

3.2 Statement of the Problem

Consider a power system with m thermal plants on the same bus supplying the load centre. Assume that the system's power demand is a known function of time over the optimization interval. The problem is to determine the generation of each plant as a function of time over the optimization interval under the following conditions:

1. The total operating cost over the optimization interval is a minimum.
2. All operating costs attributed to the fuel cost at the thermal plants (adjusted for labour, supplies and maintenance) are given by:

$$F_i(P_{s_i}(t)) = \alpha_i + \beta_i P_{s_i}(t) + \gamma_i P_{s_i}^{\ 2}(t) \qquad \$/\text{hr}$$

$$i = 1,\ldots,m \tag{3.1}$$

3. The total instantaneous power generation in the system matches the power demand.

$$P_D(t) = \sum_{i=1}^{m} P_{s_i}(t) \qquad (3.2)$$

3.3 Mathematical Formulation

The object of the optimizing computation is:

$$\text{Find} \quad \underset{P_{s_i}(t),\ i=1,\ldots,m}{\text{Min}} \int_0^{T_f} \sum_{i=1}^{m} F_i(P_{s_i}(t))dt \qquad (3.3)$$

subject to the constraint given by (3.2). Define the m dimensional control vector function $\underline{u}(t)$ by:

$$\underline{u}(t) = \text{col.}[P_{s_1}(t), P_{s_2}(t), \ldots, P_{s_m}(t)] \qquad (3.4)$$

Define next the m dimensional vectors $\underline{L}$ and $\underline{M}$ as:

$$\underline{L} = \text{col.}[\beta_1, \beta_2, \ldots, \beta_m] \qquad (3.5)$$

$$\underline{M} = \text{col.}[1, 1, \ldots, 1, 1] \qquad (3.6)$$

Finally define the mxm diagonal matrix $\underline{B}$ by:

$$\underline{B} = \text{diag}[\gamma_1, \gamma_2, \ldots, \gamma_m] \qquad (3.7)$$

Substituting (3.1) in (3.3) and using (3.4), (3.5) and (3.7) transform the cost functional of (3.3) to:

$$J_o(\underline{u}(t)) = \int_0^{T_f} \underline{L}^T\underline{u}(t)dt + \int_0^{T_f} \underline{u}^T(t)\ \underline{B}\ \underline{u}(t)dt \qquad (3.8)$$

Note that the α_i's of (3.1) are dropped from (3.8), since these are independent of the control vector $\underline{u}(t)$. Furthermore, substitution of (3.4) and (3.6) in (3.2) yields:

$$P_D(t) = \underline{M}^T\underline{u}(t) \qquad (3.9)$$

In order to cast the problem in a minimum norm formulation, the control vector $\underline{u}(t)$ is taken as an element of the Hilbert space $L_{2,\underline{B}}^{(m)}[0,T_f]$. This is the space of the m-dimensional vector valued square integrable functions defined on $[0,T_f]$endowed with the inner product defined by:

$$<\underline{V}(t), \underline{u}(t)> = \int_0^{T_f} \underline{V}^T(t)\ \underline{B}\ \underline{u}(t)dt \tag{3.10}$$

for every $\underline{V}(t)$ and $\underline{u}(t)$ in $L_{2,\underline{B}}^{(m)}[0,T_f]$. Note that the matrix $\underline{B}$ is positive definite for (3.10) to be a valid inner product definition. In the problem at hand, the γ_i's are positive which ensures the validity of (3.10).

Furthermore, the given power demand function $P_D(t)$ is considered an element of the space $L_2[0,T_f]$ of square integrable functions defined on $[0,T_f]$, with the inner product definition:

$$<x(t), y(t)> = \int_0^{T_f} x(t)y(t)dt \tag{3.11}$$

for every $x(t)$ and $y(t)$ in $L_2[0,T_f]$.

With these definitions it is seen that (3.9) defines a bounded linear transformation T: $L_{2,B}^{(m)}[0,T_f] \rightarrow L_2[0,T_f]$. Also the cost functional of (3.8) reduces to:

$$J_o(\underline{u}(t)) = <\underline{V},\underline{u}(t)> + ||\underline{u}(t)||^2 \tag{3.12}$$

where $\underline{V}$ is given by:

$$\underline{V}^T = \underline{L}^T\underline{B}^{-1} \tag{3.13}$$

Furthermore, (3.12) can be written as:

$$J_o(\underline{u}(t)) = ||\underline{u}(t) + (\underline{V}/2.)||^2 - ||\underline{V}/2.||^2 \tag{3.14}$$

Since $\underline{V}$ is independent of the control vector $\underline{u}(t)$, it is only needed to consider minimizing:

$$J_1(\underline{u}(t)) = ||\underline{u}(t) + (\underline{V}/2.)||^2 \tag{3.15}$$

Also, since a norm is a positive scalar, then minimizing the norm is equivalent to minimizing its square. Thus the final cost functional that needs to be considered is given by

$$J(\underline{u}(t)) = ||\underline{u}(t) + (\underline{V}/2.|| \tag{3.16}$$

subject to

$$P_D(t) = T(\underline{u}(t)) \tag{3.17}$$

or

$$T(\underline{u}(t)) = \underline{M}^T\underline{u}(t) \tag{3.18}$$

The problem is now formulated as a minimum norm problem whose form was given in section (2.3). The optimal solution vector $\underline{u}_\xi(t)$ is obtained in the next section.

3.4 The Optimal Solution

In view of Section 2.3, the optimal solution to the problem formulated in the previous section is given by

$$\underline{u}_\xi(t) = T^\dagger[P_D(t) + T(\underline{V}/2.)] - (\underline{V}/2.) \tag{3.19}$$

The first step in characterizing the pseudo-inverse transformation $T^\dagger$ is to obtain the adjoint transformation T^*. This is obtained by utilizing the adjoint equation:

$$\langle P_D(t),T(\underline{u}(t))\rangle = \langle T^*(P_D(t)),\underline{u}(t)\rangle \tag{3.20}$$

The left-hand side inner product is evaluated in $L_2[0,T_f]$ as

$$\langle P_D(t),T(\underline{u}(t))\rangle = \int_0^{T_f} P_D(t)[\underline{M}^T\underline{u}(t)]dt \tag{3.21}$$

The right-hand side inner product in $L^{(m)}_{2,\underline{B}}[0,T_f]$ is given by:

$$\langle T^*(P_D(t)),\underline{u}(t)\rangle = \int_0^{T_f} [T^*(P_D(t))]^T\underline{B}\ \underline{u}(t)dt \tag{3.22}$$

Thus the equality (3.20) reduces to

$$\int_{o}^{T_f} \{[T^*(P_D(t))]^T - \underline{M}^T\underline{B}^{-1}P_D(t)\}\underline{B}\ \underline{u}(t)dt = 0 \tag{3.23}$$

The equality (3.23) is satisfied for

$$[T^*(P_D(t))]^T = \underline{M}^T\underline{B}^{-1}P_D(t) \tag{3.24}$$

This yields an expression for the adjoint transformation T^* as

$$T^*[P_D(t)] = \underline{B}^{-1}\underline{M}\ P_D(t) \tag{3.25}$$

The second step is to evaluate the operator $J(P_D(t))$ using the relation (2-7), and the equations (3.9), (3.17) and (3.24) to obtain

$$J(P_D(t)) = \underline{M}^T\underline{B}^{-1}\underline{M}\ P_D(t) \tag{3.26}$$

Note that $(\underline{M}^T\underline{B}^{-1}\underline{M})$ is a quadratic form which has the scalar value

$$\underline{M}^T\underline{B}^{-1}\underline{M} = \sum_{i=1}^{m} \frac{1}{\gamma_i} \tag{3.27}$$

Thus the inverse operator J^{-1} is obtained as:

$$J^{-1}(P_D(t)) = P_D(t)/\sum_{i=1}^{m} \frac{1}{\gamma_i} \tag{3.28}$$

Finally the pseudo-inverse operator $T^\dagger$ is obtained as

$$T^\dagger[P_D(t)] = \underline{B}^{-1}\underline{M}\ P_D(t)/\sum_{i=1}^{m} \frac{1}{\gamma_i} \tag{3.29}$$

Note that substituting equations (3.6) and (3.7) in (3.29) yields:

$$T^\dagger[P_D(t)] = \text{col.}\left[\frac{P_D(t)}{\gamma_1 \sum_{i=1}^{m} \frac{1}{\gamma_i}}, \frac{P_D(t)}{\gamma_2 \sum_{i=1}^{m} \frac{1}{\gamma_i}}, \ldots, \frac{P_D(t)}{\gamma_m \sum_{i=1}^{m} \frac{1}{\gamma_i}}\right] \tag{3.30}$$

The vector $\underline{V}$ in (3.19) is given by (3.13). Substituting (3.5) and (3.7) the following is obtained:

$$\underline{V} = \text{col.}\left[\frac{\beta_1}{\gamma_1}, \frac{\beta_2}{\gamma_2}, \ldots, \frac{\beta_m}{\gamma_m}\right] \tag{3.31}$$

And using (3.18) yields:

$$T(\underline{V}/2) = \sum_{i=1}^{m} \frac{\beta_i}{2\gamma_i} \tag{3.32}$$

Thus substituting (3.30), (3.31) and (3.32) in (3.19) the optimal vector given in (3.19) is obtained. Here recalling the definition of $\underline{u}_{\xi}(t)$ in (3.4), the optimal power generations are obtained as:

$$P_{s_{\xi_i}}(t) = \{[P_D(t) + 0.5 \sum_{i=1}^{m} (\beta_i/\gamma_i)]/(\gamma_i \sum_{i=1}^{m} (1/\gamma_i))\}$$

$$- \{\beta_i/2\gamma_i\} \qquad i = 1,...,m \qquad (3.33)$$

It is worth mentioning here that the optimal solution obtained here can be shown to satisfy the optimality conditions of the classical theory of variational calculus [34].

CHAPTER IV

POWER SYSTEMS WITH HYDRO PLANTS ON SEPARATE STREAMS

4.1 Background

In this chapter the problem of obtaining optimal generation schedules for hydro-thermal electric power systems is considered. The system considered here is characterized by the presence of hydro plants that are not on the same stream. According to their order of complexity, three optimal operation problems are formulated and solved in the next three sections.

The optimization procedure is based upon predicted future system demand and a forecast of available water resources. Due to accuracy considerations present day forecasting techniques are only reliable for short time intervals. Thus the deterministic optimization problems considered here are categorized as short-range economy dispatch problems.

The inclusion of the system's transmission losses in the power balance equation is based on the well established loss formula [10]. It is noted that such a function as the transmission losses cannot be expressed in terms of only generator powers in an exact manner. There are a number of approximations involved in the loss formula derivation. However, it produces close answers with errors of up to a few percent [34]. Very sophisticated methods of calculating the loss formula coefficients exist and are being used [35, 36].

The operating costs of the hydro-electric plants do not change

with the power output. This would lead to an optimization schedule assigning the power demand to all existing hydro-plants. In practice, however, the amount of water available over the optimization interval is limited. Thus a constraint on the volume of water discharge over the optimization interval is imposed.

A brief presentation of previous investigations of hydro-thermal power systems economic operation will be given. Various optimization techniques have been used in the past. Among these are the variational calculus principles, the methods of dynamic programming and the Pontryagin's Maximum Principle.

In 1940, J. Ricard [37] obtained a set of operating schedules for a hydro-thermal system with no losses. His work was continued by W.G. Chandler, P.L. Dandeno, A.F. Glimm and L.K. Kirchmayer [38] in 1953, who included transmission losses but with constant hydraulic head. The latter method was improved in 1958 by A.F. Glimm and L.K. Kirchmayer [11] by including variable head plants. The work of Kron, who developed equivalent equations, was reported in [11].

A set of scheduling equations was developed in 1953 by R.J. Cypser [6]. These were developed under the assumption that variations in elevation and plant efficiencies can be neglected. J.J. Carey [39] suggested an approach which will linearize Cypser's equations. C.W. Watchorn gives in [4] a set of equations to be satisfied in order to achieve maximum economy.

It should be noted that the above mentioned investigations employed the Euler equation of the calculus of variations to obtain the scheduling equations.

The work of R.A. Arismunander [40] in 1960 and later in 1962 [41] jointly with F. Noakes dealt with short-range optimization of hydro-thermal systems. He employed all the necessary and sufficient conditions for optimality of the variational calculus. In addition to this he proved the equivalence of all previously developed equations.

In 1962, J.H. Drake, L.K. Kirchmayer, R.B. Mayall and H. Wood [7] presented a dispatch formula based on the calculus of variations. This formula is restricted to the case where all the hydro-plants operate with constant head. The system considered has series plants, multiple chains of plants and intermediate reservoirs.

L.K. Kirchmayer and R.J. Ringlee presented a dispatch formula for a hydro-thermal power system in[42]. Head variations were considered. The formula applies for power systems having one hydro-plant.

In a discussion of the paper in reference [7], C.W. Watchorn points out the importance of considering variable head for the optimization of such systems. In separate discussions of the same paper, C.W. Watchorn and R.A. Arismunandar, both point out that a river time delay of a couple of hours is highly important for accurate optimization of many power systems.

B. Bernholtz and L.V. Graham [43] presented a dynamic programming solution to the hydro-thermal optimization problem. L.K. Kirchmayer and R.J. Ringlee [42] have also a presentation of some results and conclusions with regard to dynamic programming solution.

The application of both the Pontryagin's Maximum Principle and Dynamic Programming to the hydro-thermal dispatch problem was considered by E.B. Dahlin in 1964 [5] and later, jointly with D.W.C. Shen [44].

The general dispatch formulas obtained were applicable to a wide class of systems. These were the systems with hydro-plants having fixed head, varying head, and hydraulic coupling, both with and without river transport delay.

The economic operation of a simplified model system and the long range operation of a multi-reservoir system is considered by I. Hano, Y. Tamura and S. Narita in 1966 [3]. They employed the Pontryagin's Maximum Principle to obtain the scheduling equations.

The various factors involved in hydro-thermal coordination and their interrelation required for optimum generation are discussed by C.W. Watchorn in 1967 [3]. In a discussion of C.W. Watchorn's paper, G.S. Christensen points out that the scheduling equations obtained constitute only a necessary condition for optimality. He suggested the steepest descent method to search for the global optimum mode of operation.

4.2 Power System with Fixed Head Hydro-Plants and Negligible Transmission Losses

In this section, a hydro-thermal electric power system is considered. It is assumed that the transmission losses and head variations at the hydro plants are negligible. Imposing a constraint on the volume of water discharged over the optimization interval is equivalent to constraining the hydro-energy over the same interval. The formulation of the problem under this condition will fail to define the optimal hydraulic generations. This difficulty arises if one uses either of

the variational principles or the functional analytic technique adopted in this thesis. The main reason for this is the absence of second order terms of the hydro-generations in both the cost functional and the constraints. To avoid this difficulty in this particular problem, it is assumed here that the integral of the square of the hydro generation over the optimization interval is a prespecified constant [45]. This implies the presence of an upperbound on the hydro-energy available.

4.2.1 Statement of the Problem

A hydro-thermal electric power system is considered. The system has n generating plants, of which m are thermal. A prediction of the system's future power demand is assumed available. The problem is to determine the generation of each plant in order to minimize the operating cost of the system under the following conditions:

1. The operating costs at the ith thermal plant is approximated by:

$$F_i[P_{s_i}(t)] = \alpha_i + \beta_i P_{s_i}(t) + \gamma_i P_{s_i}^{\,2}(t) \qquad \$/hr \qquad (4.2.1)$$

2. The total generation in the system matches the load. Transmission losses may be neglected.

3. The integral of the square of the hydro generation over the optimization interval is a prespecified constant.

4.2.2A Minimum Norm Formulation

The object of the optimizing computation is

$$\text{find} \quad \underset{\substack{P_{s_i}(t) \\ i=1,\ldots,m}}{\text{Min}} \int_0^{T_f} \sum_{i=1}^{m} \{\alpha_i + \beta_i P_{s_i}(t) + \gamma_i P_{s_i}^{\,2}(t)\}dt \qquad (4.2.2)$$

subject to the constraints:

$$P_D(t) = \sum_{i=1}^{m} P_{s_i}(t) + \sum_{i=m+1}^{n} P_{h_i}(t) \tag{4.2.3}$$

$$\int_{o}^{T_f} P_{h_i}^{\;2}(t)dt = K_i \qquad i = m+1,\ldots,n \tag{4.2.4}$$

The constraints given by (4.2.4) may be included in the cost functional defined in (4.2.2). The resulting augmented cost functional is then given by:

$$J_o(P_{s_1}(t),\ldots,P_{h_n}(t)) = \int_{o}^{T_f}\left\{\left[\sum_{i=1}^{m} \beta_i P_{s_i}(t) + \gamma_i P_{s_i}^{\;2}(t)\right] + \left[\sum_{i=m+1}^{n} k_i P_{h_i}^{\;2}(t)\right]\right\}dt \tag{4.2.5}$$

Here terms explicitly independent of the power generations are neglected in the cost functional. The k_i's are unknowns to be determined such that (4.2.4) is satisfied.

Define the nx1 column vector control function $\underline{u}(t)$ as:

$$\underline{u}(t) = \text{col.}[P_{s_1}(t), P_{s_2}(t), \ldots, P_{s_m}(t), P_{h_{m+1}}(t), \ldots, P_{h_n}(t)] \tag{4.2.6}$$

Define two nx1 column vectors $\underline{L}$ and $\underline{M}$ as

$$\underline{L} = \text{col.}[\beta_1, \ldots, \beta_m, 0, \ldots, 0] \tag{4.2.7}$$

$$\underline{M} = \text{col.}[1, \ldots, 1] \tag{4.2.8}$$

and an nxn diagonal matrix $\underline{B}$ by:

$$\underline{B} = \text{diag}[\gamma_1, \gamma_2, \ldots, \gamma_m, k_{m+1}, \ldots, k_n] \tag{4.2.9}$$

Using these definitions in (4.2.5) and (4.2.3), the problem

reduces to minimizing

$$J_o(\underline{u}(t)) = \int_0^{T_f} \{\underline{L}^T\underline{u}(t) + \underline{u}^T(t)\underline{B}\ \underline{u}(t)\}dt \tag{4.2.10}$$

subject to satisfying

$$P_D(t) = \underline{M}^T\underline{u}(t) \tag{4.2.11}$$

The control vector function $\underline{u}(t)$ is considered an element of the Hilbert space $L^n_{2,\underline{B}}[0,T_f]$. This is the space of all n-dimensional vector valued square integrable functions defined on the interval $[0,T_f]$. The space is endowed with the inner product definition

$$<\underline{V}(t),\underline{u}(t)> = \int_0^{T_f} \underline{V}(t)\underline{B}\ \underline{u}(t)dt \tag{4.2.12}$$

for every $\underline{V}(t)$ and $\underline{u}(t)$ in the space $L^n_{2,B}[0,T_f]$. The power demand function $P_D(t)$ is considered to belong to the Hilbert space $L_2[0,T_f]$ of the single valued square integrable functions defined on the interval $[0,T_f]$. The inner product definition for this space is:

$$<\underline{x}(t),\underline{y}(t)> = \int_0^{T_f} x(t)y(t)dt \tag{4.2.13}$$

for every x(t) and y(t) in $L_2[0,T_f]$.

The cost functional given by (4.2.10) then reduces to

$$J_o(\underline{u}(t)) = <\underline{V},\underline{u}(t)> + ||\underline{u}(t)||^2 \tag{4.2.14}$$

where

$$\underline{V}^T = \underline{L}^T\underline{B}^{-1} \tag{4.2.15}$$

The constraint given by (4.2.11) is seen to define a bounded linear transformation T: $L^n_{2,\underline{B}}[0,T_f]\rightarrow L_2[0,T_f]$ of the form

$$P_D(t) = T[\underline{u}(t)] \tag{4.2.16}$$

Following the same procedure as in Section (3.2.2), the problem

reduces to minimizing:

$$J(\underline{u}(t)) = ||\underline{u}(t) + V/2|| \tag{4.2.17}$$

subject to

$$P_D(t) = T[\underline{u}(t)] \tag{4.2.18}$$

This is a minimum norm problem of the form discussed in Chapter 2.

4.2.3 The Optimal Solution

According to the results cited in Chapter 2, the optimal solution to the problem formulated in Section 4.2.2. is given by:

$$\underline{u}_\xi(t) = T^\dagger[P_D(t) + T(\underline{V}/2)] - (\underline{V}/2) \tag{4.2.19}$$

The system's transformation T here is the same as that of the all-thermal problem solved in Chapter 3. Moreover, the spaces involved are essentially the same. Thus the derivation of the optimal solution as given in (4.2.19) follows essentially the same steps as given in Section (3.2.3). The optimal generations are thus found to be:

$$P_{s_{i_\xi}}(t) = \frac{[P_D(t) + \frac{1}{2}\sum_{i=1}^{m}\frac{\beta_i}{\gamma_i}]}{\gamma_i[\sum_{i=1}^{m}\frac{1}{\gamma_i} + \sum_{i=m+1}^{n}\frac{1}{k_i}]} - \frac{\beta_i}{2\gamma_i} \tag{4.2.20}$$

$$i = 1,\ldots,m$$

$$P_{h_{i_\xi}}(t) = \frac{[P_D(t) + \frac{1}{2}\sum_{i=1}^{m}\frac{\beta_i}{\gamma_i}]}{k_i[\sum_{i=1}^{m}\frac{1}{\gamma_i} + \sum_{i=m+1}^{n}\frac{1}{k_i}]} \tag{4.2.21}$$

$$i = m+1,\ldots,n$$

It can be seen that the sum of the optimal generations as given by (4.2.20) and (4.2.21) yields the power demand function $P_D(t)$. This

is in agreement with the constraint given by (4.2.3). The (n-m) unknown k_i's can be obtained by substituting (4.2.21) in the corresponding constraint (4.2.4).

4.3 Power System Containing Hydro Plants with Negligible Head Variations

The problem posed and solved in this section is the same as the problem in Section 4.2, except that the transmission losses in the system are also included here. The transmission losses are assumed to be represented by the General Loss Formula. The General Loss Formula provides second order powers of the hydro-generations, which makes it possible to take directly the hydro-water draw-down constraint into consideration.

The results of this section were first reported in [46]. It is easily shown that the optimal generations obtained in this section satisfy the variational conditions for optimality.

4.3.1 Statement of the Problem

Consider a power system with m thermal plants and (n-m) hydro plants. A prediction of the system's future load and water supply is assumed available for the optimization interval under consideration. The problem is to find the active power generation of each plant as a function of time over the optimization interval under the following conditions:

1. The total operating cost of the thermal plants over the optimization interval is a minimum.

2. The operating costs at the ith thermal plant are approximated by:

$$F_i(P_{s_i}(t)) = \alpha_i + \beta_i P_{s_i}(t) + \gamma_i P_{s_i}^{2}(t) \qquad \$/\text{hr} \qquad (4.3.1)$$

3. The total active generation in the system matches the load

plus the transmission losses.

4. The time integral of water discharge from each hydro plant is a prespecified constant amount.

4.3.2 A Minimum Norm Formulation

The object of the optimizing computation is

$$\underset{\substack{P_{s_i}(t)\\ i=1,\dots,m}}{\text{Min}} \int_0^{T_f} \sum_{i=1}^{m} F_i(P_{s_i}(t))dt \tag{4.3.2}$$

The generation schedule sought must satisfy the active power balance equation:

$$P_D(t) = \sum_{i=1}^{m} P_{s_i}(t) + \sum_{i=m+1}^{n} P_{h_i}(t) - P_L(t) \tag{4.3.3}$$

The transmission loss is a quadratic function of the active power generated by the system plants and is given by [10]:

$$P_L(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} P_i(t)B_{ij}P_j(t) + \sum_{i=1}^{n} B_{io}P_i(t) + K_{Lo} \tag{4.3.4}$$

with B_{ij}'s and B_{io}'s and K_{Lo} being the loss formula coefficients which are assumed to be known, with the property:

$$B_{ij} = B_{ji} \qquad (i,j = 1, \dots, n) \tag{4.3.5}$$

Furthermore, the water discharge at each hydro plant must satisfy the following constraint on the volume of water used during the optimization interval:

$$\int_0^{T_f} q_i(t)dt = b_i \qquad i = m+1,\dots,n \tag{4.3.6}$$

The ith hydro plant's active power generation $P_{h_i}(t)$ is given by:

$$P_{h_i}(t) = \frac{\eta_i q_i(t) h_i(t)}{11.8} \quad \text{KW} \tag{4.3.7}$$

Assuming constant head and efficiency at each hydro plant, the constraint given by (4.3.6) is equivalent to the following energy constraint:

$$\int_0^{T_f} P_{h_i}(t)dt = K_i \qquad i = m+1,\ldots,n \tag{4.3.8}$$

with

$$K_i = \frac{\eta_i h_i b_i}{11.8} \tag{4.3.9}$$

The constraint given by equation (4.3.3) can be added to the integrand of the cost functional given by (4.3.2) using the unknown function $\theta(t)$, so that a modified cost functional is obtained:

$$J_o(P_{s_i}(t), P_{h_i}(t)) = \int_0^{T_f} [\sum_{i=1}^{m} F_i(P_{s_i}(t)) + \theta(t)(P_D(t) - \sum_{i=1}^{m} P_{s_i}(t) - \sum_{i=m+1}^{n} P_{h_i}(t) + P_L(t))]dt \tag{4.3.10}$$

Substituting for $P_L(t)$ from (4.3.4) and $F_i(P_{s_i}(t))$ from (4.3.1), (4.3.10) can be written as:

$$J_1(P_{s_i}(t), P_{h_i}(t)) = \int_0^{T_f} [\sum_{i=1}^{m} (\beta_i + \theta(t)[B_{io} - 1])P_{s_i}(t) + \sum_{i=m+1}^{n} \theta(t)[B_{io} - 1]\, P_{h_i}(t) + \sum_{i=1}^{m} \gamma_i P_{s_i}{}^2(t) + \theta(t) \sum_{i=1}^{n} \sum_{j=1}^{n} P_i(t) B_{ij} P_j(t)]dt \tag{4.3.11}$$

Here terms explicitly independent of the power generations are dropped.

Define the nx1 column vector :

$$\underline{u}(t) = \text{col.}[P_{s_1}(t), P_{s_2}(t), \ldots, P_{h_{m+1}}(t), \ldots, P_{h_n}(t)] \tag{4.3.12}$$

$$\underline{L}(t) = \text{col.}[(\beta_1 + C_1\theta(t)), \ldots, C_{m+1}\theta(t), \ldots, C_n\theta(t)] \tag{4.3.13}$$

where

$$C_i = B_{io} - 1 \qquad i = 1, \ldots, n \tag{4.3.14}$$

And the symmetric (nxn) matrix

$$\underline{B}(t) = (b_{ij}(t)) \tag{4.3.15}$$

with

$$b_{ij}(t) = B_{ij}\theta(t) \qquad i,j = 1, \ldots, n \ (i \neq j)$$

$$b_{ii}(t) = \gamma_i + B_{ii}\theta(t) \qquad i = 1, \ldots, m$$

$$b_{ii}(t) = B_{ii}\theta(t) \qquad i = m+1, \ldots, n$$

then (4.3.11) becomes:

$$J_1(\underline{u}(t)) = \int_0^{T_f} \{\underline{L}^T(t)\underline{u}(t) + \underline{u}^T(t)\underline{B}(t)\underline{u}(t)\}dt \tag{4.3.16}$$

If we let

$$\underline{V}(t) = \underline{B}^{-1}\underline{L}(t) \tag{4.3.17}$$

equation (4.3.16) can be written as:

$$J_1(\underline{u}(t)) = \int_0^{T_f} \{\underline{V}^T(t)\underline{B}(t)\underline{u}(t) + \underline{u}^T(t)\underline{B}(t)\underline{u}(t)\}dt \tag{4.3.18}$$

or

$$J_1(\underline{u}(t)) = \int_0^{T_f} [\{(\underline{u}(t) + \frac{\underline{V}(t)}{2})^T \underline{B}(t)(\underline{u}(t) + \frac{\underline{V}(t)}{2})\} - (\frac{\underline{V}^T(t)}{2}\, \underline{B}(t)\, \frac{\underline{V}(t)}{2})]dt \tag{4.3.19}$$

The last term in the integrand of equation (4.3.19) does not depend explicitly on $\underline{u}(t)$, so that it is necessary only to consider:

$$J(\underline{u}(t)) = \int_0^{T_f} \{(\underline{u}(t) + \frac{\underline{V}(t)}{2})^T \underline{B}(t)(\underline{u}(t) + \frac{\underline{V}(t)}{2})\}dt \tag{4.3.20}$$

The problem is now transformed to that of finding a control vector $\underline{u}(t)$, that minimizes the cost functional given by (4.3.20) while satisfying the energy constraint given by (4.3.8). Notice that $\theta(t)$ will be determined so that the optimal control vector satisfies the active power balance equation (4.3.3).

The control vector $\underline{u}(t)$ is considered to be an element of the Hilbert space $L^n_{2,\underline{B}}[0,T_f]$ of the n-vector valued square integrable functions defined on $[0,T_f]$ whose inner product is given by

$$<\underline{V}(t),\underline{u}(t)> = \int_0^{T_f} \underline{V}^T(t)\underline{B}(t)\underline{u}(t)dt \tag{4.3.21}$$

for every $\underline{V}(t)$ and $\underline{u}(t)$ in $L^{(n)}_{2,\underline{B}}[0,T_f]$, provided that $\underline{B}(t)$ is positive definite. This means that $\underline{u}_\xi(t)\varepsilon L^n_{2,B}[0,T_f]$ which minimizes:

$$J(\underline{u}(t)) = ||\underline{u}(t) + \frac{\underline{V}(t)}{2}||^2 \tag{4.3.22}$$

and satisfies (4.3.8) is sought.

Define the (n-m)x1 column vector.

$$\underline{\xi} = \text{col.}[K_{m+1},\ldots,K_n] \tag{4.3.23}$$

and the nx(n-m) matrix

$$\underline{M} = \text{col.}[\underline{0},\underline{I}] \tag{4.3.24}$$

with $\underline{0}$ being the mx(n-m) matrix whose elements are all zeros, and $\underline{I}$ is the (n-m)x(n-m) identity matrix, so that the constraints of equation (4.3.8) can be expressed as:

$$\underline{\xi} = \int_0^{T_f} \underline{M}^T\underline{u}(t)dt \tag{4.3.25}$$

Equation (4.3.25) can be shown to define a bounded linear transformation $T: L^n_{2,B}[0,T_f] \to R^{(n-m)}$.

The real vector space $R^{(n-m)}$ is endowed with the Euclidean inner product definition:

$$\langle \underline{X}, \underline{Y} \rangle = \underline{X}^T \underline{Y} \tag{4.3.26}$$

for every $\underline{X}$ and $\underline{Y}$ in $R^{(n-m)}$.

Equation (4.3.25) is now written as:

$$\underline{\xi} = T[\underline{u}(t)] \tag{4.3.27}$$

with $T[\underline{u}(t)]$ given by:

$$T[\underline{u}(t)] = \int_0^{T_f} \underline{M}^T \underline{u}(t)dt \tag{4.3.28}$$

The problem under consideration has been reduced to finding the optimal generation vector $\underline{u}(t)$ which minimizes the cost functional given by (4.3.22) such that (4.3.27) is satisfied for the given vector $\underline{\xi}$.

4.3.3. The Optimal Solution.

In view of the results cited in Chapter 2, there is exactly one optimal solution to the problem formulated in the previous section, namely the optimal vector.

$$\underline{u}_{\xi}(t) = T^{\dagger}[\underline{\xi} + T(\frac{\underline{V}(t)}{2})] - \frac{\underline{V}(t)}{2} \tag{4.3.29}$$

where $T^{\dagger}$ is obtained as follows:

T^*, the adjoint of T, is obtained by using the identity:

$$\langle \underline{\xi}, \underline{Tu}(t) \rangle_{R^{(n-m)}} = \langle \underline{T^* \xi}, \underline{u}(t) \rangle_{L^{(n)}_{2,\underline{B}}[0,T_f]} \tag{4.3.30}$$

where in $R^{(n-m)}$ we have:

$$\langle \underline{\xi}, \underline{Tu}(t) \rangle = \underline{\xi}^T \underline{Tu}(t) \tag{4.3.31}$$

using (4.3.28), the following is obtained:

$$<\underline{\xi},\underline{Tu}(t)> = \int_0^{T_f} \underline{\xi}^T\underline{M}^T\underline{u}(t)dt \qquad (4.3.32)$$

or

$$<\underline{\xi},\underline{Tu}(t)> = \int_0^{T_f} (\underline{B}^{-1}(t)\underline{M\xi})^T\underline{B}(t)\underline{u}(t)dt \qquad (4.3.33)$$

where use is made of

$$B^{-1}(t) = (\underline{B}^{-1}(t))^T \qquad (4.3.34)$$

But in $L_{2,B}^{(n)}$ $[0,T_f]$ we have:

$$<\underline{B}^{-1}(t)\underline{M\xi},\underline{u}(t)> = \int_0^{T_f} (\underline{B}^{-1}(t)\underline{M\xi})^T\underline{B}(t)\underline{u}(t)dt \qquad (4.3.35)$$

hence, by equation (4.3.30) it is seen that

$$\underline{T^*\xi} = \underline{B}^{-1}(t)\underline{M\xi} \qquad (4.3.36)$$

which defines the adjoint of T.

Next we find the transformation J given by:

$$\underline{J\xi} = T[T^*\xi] \qquad (4.3.37)$$

using equations (4.3.28) and (4.3.36) as

$$\underline{J\xi} = [\int_0^{T_f}\underline{M}^T\underline{B}^{-1}(t)\underline{M}dt]\xi \qquad (4.3.38)$$

which yields

$$\underline{J}^{-1}[\xi] = [\int_0^{T_f} \underline{M}^T\underline{B}^{-1}(t)\underline{M}\, dt]^{-1}\underline{\xi} \qquad (4.3.39)$$

Finally, using the definition of the pseudo-inverse transformation $T^\dagger$, one obtains:

$$\underline{T^\dagger\xi} = \underline{B}^{-1}\underline{M}[\int_0^{T_f} \underline{M}^T\underline{B}^{-1}(t)\underline{M}\, dt]^{-1}\underline{\xi} \qquad (4.3.40)$$

Thus the optimal generation vector $\underline{u}_\xi(t)$ is given by:

$$\underline{u}_{\xi}(t) = \underline{B}^{-1}(t)\underline{M}[\int_{0}^{T_f} \underline{M}^T\underline{B}^{-1}(t)\underline{M}\, dt]^{-1}\{\underline{\xi} + \frac{1}{2}\int_{0}^{T_f}\underline{M}^T\underline{V}(t)dt\} - \frac{\underline{V}(t)}{2} \tag{4.3.41}$$

The optimal solution obtained here may be written in component form. However, this cannot be done unless the number of plants is specified and the inverse of $\underline{B}(t)$ is evaluated. The optimal solution contains the unknown function $\theta(t)$ which is determined such that the power balance equation is satisfied.

4.3.4 Implementing the Optimal Solution.

Without loss of generality, consider a power system whose loss formula contains zero off-diagonal coefficients, then $\underline{B}(t)$ can be written in partitioned form as:

$$\underline{B}(t) = \text{diag}[\underline{B}_{ss}(t), \underline{B}_{HH}(t)] \tag{4.3.42}$$

with

$$\underline{B}_{ss}(t) = \text{diag}[(\gamma_1 + \theta(t)B_{11}),\ldots,(\gamma_m + \theta(t)B_{mm})] \tag{4.3.43}$$

Let

$$\underline{L}(t) = \text{col.}[\underline{L}_{ss}(t), \underline{L}_{HH}(t)] \tag{4.3.44}$$

with

$$\underline{L}_{ss}(t) = \text{col.}[(\beta_1 + \theta(t)C_1),\ldots,(\beta_m + \theta(t)C_m)] \tag{4.3.45}$$

$$\underline{L}_{HH}(t) = \text{col.}[\theta(t)C_{m+1},\ldots,\theta(t)C_n] \tag{4.3.46}$$

Substituting in equation (4.3.41), the optimal solution for this particular system is obtained as

$$\underline{u}_{\xi}(t) = \frac{1}{\theta(t)\int_{0}^{T}\theta^{-1}(t)\, dt}\left[\frac{0}{\underline{R}}\right] - \frac{1}{2}\left[\frac{\underline{B}_{ss}^{-1}\,\underline{L}_{ss}}{\underline{B}_{HH}^{-1}\,\underline{L}_{HH}}\right] \tag{4.3.47}$$

where

$$\underline{R} = \text{col.}[(K_{m+1} + \frac{C_{m+1}T}{2B_{(m+1)(m+1)}}), \ldots,$$

$$(K_n + \frac{C_n T}{2B_{nn}})] \qquad (4.3.48)$$

so that the optimal power generations are obtained as:

$$P_{s_{i_\xi}}(t) = -\frac{[\beta_i + C_i\ \theta(t)]}{2[\gamma_i + B_{ii}\ \theta(t)]} \qquad i = 1,\ldots,m \qquad (4.3.49)$$

$$P_{h_{i_\xi}}(t) = \frac{1}{\theta(t)\int_0^{T_f} \theta^{-1}(t)\ dt}[K_i + \frac{C_i\ T_f}{2B_{ii}}] - \frac{C_i}{2B_{ii}}$$

$$i = m+1,\ldots,n \qquad (4.3.50)$$

The last expression contains the unknown function $\theta(t)$ which will be determined so that the power balance equation (4.3.3) is satisfied.

Using equation (4.3.4), equation (4.3.3) is written as:

$$P_D(t) + \sum_{i=1}^{m} B_{ii}\ P_{s_{i_\xi}}^{\,2}(t) + \sum_{i=m+1}^{n} B_{ii}\ P_{h_{i_\xi}}^{\,2}(t) +$$

$$\sum_{i=1}^{m} C_i\ P_{s_{i_\xi}}(t) + \sum_{i=m+1}^{n} C_i\ P_{h_{i_\xi}}(t) + K_{Lo} = 0 \qquad (4.3.51)$$

then substituting in (4.3.51) for $P_{s_{i_\xi}}(t)$ and $P_{h_{i_\xi}}(t)$ as given by (4.3.49) and (4.3.50) respectively we get

$$\frac{x^2(t)D}{[\int_0^{T_f} x(t)dt]^2} + \sum_{i=1}^{m} \frac{A_2^{(i)}\ x^2(t) + A_1^{(i)}\ x(t) + A_o^{(i)}}{B_2^{(i)}\ x^2(t) + B_1^{(i)}\ x(t) + B_o^{(i)}} = y(t)$$

$$(4.3.52)$$

where

$$x(t) = \theta^{-1}(t)$$

$$D = \sum_{i=m+1}^{n} B_{ii}\, m_i^{2}$$

$$m_i = K_i + \frac{C_i\, T_f}{2B_{ii}} \qquad i = m+1,\ldots,n$$

$$y(t) = -[a + P_D(t)]$$

$$a = K_{Lo} - \sum_{i=m+1}^{n} \frac{C_i^{2}}{4B_{ii}}$$

$$A_2^{(i)} = b_i\, \beta_i \qquad i = 1,\ldots,m$$

$$A_1^{(i)} = C_i\, b_i - \beta_i\, e_i \qquad i = 1,\ldots,m$$

$$A_o^{(i)} = -e_i\, C_i \qquad i = 1,\ldots,m$$

$$b_i = B_{ii}\, \beta_i - 2C_i\, \gamma_i \qquad i = 1,\ldots,m$$

$$e_i = C_i\, B_{ii} \qquad i = 1,\ldots,m$$

$$B_2^{(i)} = 4\gamma_i^{2} \qquad i = 1,\ldots,m$$

$$B_1^{(i)} = 8\gamma_i\, B_{ii} \qquad i = 1,\ldots,m$$

$$B_o^{(i)} = 4B_{ii}^{2} \qquad i = 1,\ldots,m$$

Thus equations (4.3.49), (4.3.50) and (4.3.52) completely define the optimal generation vector required.

4.3.5 A Computational Example

The method which is used to find the optimal solution is a scanning process that searches for a function $x(t)$ which is a solution to (4.3.52) and simultaneously yields a positive definite matrix $\underline{B}(t)$ (which makes the definition of the space $L_{2,\underline{B}}^{(n)}[0,T_f]$ valid). Furthermore, the optimal

generations must be physically realizable in the sense that no negative or complex active power generations are permissible.

The procedure followed in solving (4.3.52) is one of discretization which will result in a system of N simultaneous algebraic equations in N unknowns $(x_1,\ldots,x_N)$ which can be solved by any classical iteration technique.

A study was made of a sample system with the following particulars:

Number of thermal plants (m) = 1.

Number of hydro-plants (n-m) = 2.

Loss formula coefficients

$B_{11} = 1.6000 \times 10^{-4}$ $\quad B_{22} = 2.2000 \times 10^{-4}$ $\quad B_{33} = 1.6000 \times 10^{-4}$

$B_{10} = 0.00$ $\quad B_{20} = 0.00$ $\quad B_{30} = 0.00$

Thermal Plant's cost equation:

$$F(P_s(t)) = \alpha + 4.0\,P_s(t) + 1.20 \times 10^{-2}\,P_s^{\,2}(t) \quad \$/\text{Hr.}$$

The energy constraints on the hydro-plants:

$K_2 = 3600.00$ MW-Hr.

$K_3 = 2400.00$ MW-Hr.

Example (1):

Let

$$P_D(t) = 400.00 \text{ MW} \qquad 0 \le t \le T_f \qquad T_f = 24 \text{ Hrs.}$$

then equation (4.3.52) reduces to a second order algebraic equation in x(t), which yields a feasible x(t) given by:

$$x(t) = 0.21629 \qquad 0 \le t \le T_f$$

so that we get the following generation allocation by using equations (4.3.49) and (4.3.50):

$$\text{Optimal thermal power generation } P_s(t) = 160.68 \quad MW(0 \le t \le T_f)$$

$$\text{Optimal hydro-power generation } P_{h_2}(t) = 150.00 \quad MW(0 \le t \le T_f)$$

$$P_{h_3}(t) = 100.00 \quad MW(0 \le t \le T_f)$$

The result of this example agrees with the obvious assumption of constant, hydro-generations at the average values satisfying the energy constraints and the value of thermal power generation that satisfies the power balance equation.

Example (2):

Let

$$P_D(t) = 400.00 \qquad MW(0 \le t \le \frac{T_f}{2})$$

$$= 600.00 \qquad MW(\frac{T_f}{2} < t \le T_f)$$

$$T_f = 24 \text{ Hrs.}$$

then equation (4.3.52) reduces to two simultaneous algebraic equations in two unknowns x_1 and x_2 where

$$x(t) = x_1 \qquad 0 \le t \le \frac{T_f}{2}$$

$$= x_2 \qquad \frac{T_f}{2} < t \le T_f$$

The values of x_1 and x_2 satisfying these two equations and which simultaneously yield a feasible optimal power generation are found to be:

$$x_1 = 0.20008$$

$$x_2 = 0.19349$$

with the corresponding optimal power generations given by:

$$P_s(t) = 249.53 \text{ MW}, P_{H_2}(t) = 114.42 \text{ MW}, P_{H_3}(t) = 49.40 \text{ MW}$$
$$0 \le t \le \frac{T_f}{2}$$

$$P_s(t) = 288.21 \text{ MW}, P_{H_2}(t) = 185.58 \text{ MW}, P_{H_3}(t) = 150.70 \text{ MW}$$
$$\frac{T_f}{2} < t \le T_f$$

4.4 Power System with Variable Head Hydro-Plants and Transmission Losses

The problem considered in this section is more general than the ones dealt with in the previous sections. The results obtained in this section are shown to agree with Kron's results [11], that is, for the case of one thermal and one hydro plant system. The work of this section was reported by the author jointly with G.S. Christensen in [47-48].

4.4.1 Statement of the Problem.

Consider an electric power system with m thermal plants and (n-m) hydro-plants. A prediction of the system's future power demand and water supply is assumed available for the optimization interval. The problem is to find the active power generation of each plant as a function of time under the following conditions:

1. The total operating costs of the thermal plants over the optimization interval is a minimum.

2. The operating costs at the ith thermal plant are approximated by:

$$F_i[P_{s_i}(t)] = \alpha_i + \beta_i P_{s_i}(t) + \gamma_i P_{s_i}^2(t) \quad \$/\text{Hr.} \tag{4.4.1}$$

3. The total active generation in the system matches the load plus

the losses.

4. The transmission losses in the system may be represented by the general loss formula.

5. The time integral of water discharge for each hydro-plant is a prespecified constant amount.

6. The ith hydro-plant's reservoir is assumed to have vertical sides and small capacity. Head variations are related to the discharge by the reservoir's dynamic equation.

$$S_i \dot{h}_i(t) = i_i(t) - q_i(t) \tag{4.4.2}$$

7. The tail-race elevation at any of the hydro-plants does not change with the water discharge.

4.4.2 A Minimum Norm Formulation

The object of the optimizing computation is:

$$\underset{\substack{P_{s_i}(t) \\ i=1,\ldots,m}}{\text{Min}} \int_0^{T_f} \sum_{i=1}^{m} F_i[P_{s_i}(t)]dt \tag{4.4.3}$$

The generation schedule sought must satisfy the active power balance equation:

$$P_D(t) = \sum_{i=1}^{m} P_{s_i}(t) + \sum_{i=m+1}^{n} P_{h_i}(t) - P_L(t) \tag{4.4.4}$$

The transmission power loss is given by:

$$P_L(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} P_i(t)\, B_{ij}\, P_j(t) + \sum_{i=1}^{n} B_{io}\, P_i(t) + K_{Lo} \tag{4.4.5}$$

Furthermore, the water discharge at each hydro plant is to satisfy the following constraint on the volume of water used over the optimization interval:

$$\int_{0}^{T_f} q_i(t)dt = b_i \qquad i = m+1,\ldots,n \qquad (4.4.6)$$

The ith hydro plant's active power generation $P_{h_i}(t)$ is given by

$$P_{h_i}(t) = \frac{q_i(t)h_i(t)}{G_i} \qquad i = m+1,\ldots,n \qquad (4.4.7)$$

with

$$G_i = 11.8/\eta_i$$

The effective hydraulic head at the ith hydro plant $h_i(t)$ may be expressed using (4.4.2) as:

$$h_i(t) = \frac{1}{S_i}\left[\int_0^t i_i(\sigma)d\sigma - \int_0^t q_i(\sigma)d\sigma\right] + h_i(0) \qquad (4.4.8)$$

$$i = m+1,\ldots,n$$

Thus substituting (4.4.8) in (4.4.7), $h_i(t)$ is eliminated. The expression for the hydro-power is then given by:

$$P_{h_i}(t) = q_i(t)\left[\frac{h_i(o)}{G_i} + \frac{1}{S_iG_i}\int_0^t i_i(\sigma)d\sigma - \frac{1}{S_iG_i}\int_0^t q_i(\sigma)d\sigma\right]$$

$$i = m+1,\ldots,n \qquad (4.4.9)$$

Define the following quantities:

$$N_i(t) = \frac{h_i(o)}{G_i} + \frac{1}{S_iG_i}\int_0^t i_i(\sigma)d\sigma \qquad i = m+1,\ldots,n \qquad (4.4.10)$$

$$E_i = (S_iG_i)^{-1} \qquad i = m+1,\ldots,n \qquad (4.4.11)$$

then (4.4.9) reduces to:

$$P_{h_i}(t) = q_i(t)\left[N_i(t) - E_i\int_0^t q_i(\sigma)d\sigma\right] \qquad i = m+1,\ldots,n \qquad (4.4.12)$$

It is convenient to introduce the volume of water discharges variable $Q_i(t)$ by the following definition:

$$Q_i(t) = \int_0^t q_i(\sigma)d\sigma \qquad i = m+1,\ldots,n \tag{4.4.13}$$

thus

$$q_i(t) = \dot{Q}_i(t) \qquad i = m+1,\ldots,n \tag{4.4.14}$$

This reduces the power generation as given by (4.4.12) to

$$P_{h_i}(t) = \dot{Q}_i(t)[N_i(t) - E_i\, Q_i(t)] \qquad i = m+1,\ldots,n \tag{4.4.15}$$

And the constraint on the volume of water discharged over the optimization interval given by (4.4.6) reduces to:

$$Q_i(T_f) = b_i \qquad i = m+1,\ldots,n \tag{4.4.16}$$

The problem now is to minimize the cost functional given by (4.4.3) subject to satisfying (4.4.4), (4.4.5),(4.4.15) and (4.4.16). Including (4.4.4), (4.4.5) and (4.4.15) in the cost functional yields:

$$\begin{aligned} J_o(P_{s_i}(t),P_{h_i}(t),Q_i(t)) = \int_0^{T_f} \Big\{ & \sum_{i=1}^{m} F_i[P_{s_i}(t)] + \lambda(t)[P_D(t) \\ & + \sum_{i=1}^{n} \sum_{j=1}^{n} P_i(t)\, B_{ij}\, P_j(t) + \sum_{i=1}^{n} B_{io}P_i(t) \\ & + K_{Lo} - \sum_{i=1}^{m} P_{s_i}(t) - \sum_{i=m+1}^{n} P_{h_i}(t)] \\ & + \sum_{i=m+1}^{n} n_i(t)[N_i(t)\dot{Q}_i(t) - E_iQ_i(t)\dot{Q}_i(t) \\ & - P_{h_i}(t)]\Big\}dt \end{aligned} \tag{4.4.17}$$

Using the following relations:

$$\int_0^{T_f} n_i(t)\dot{Q}_i(t)Q_i(t)dt = \tfrac{1}{2}[n_i(T_f)b_i^2 - \int_0^{T_f} n_i(t)Q_i^2(t)dt]$$

$$i = m+1,\ldots,n \qquad (4.4.18)$$

$$\int_0^{T_f} n_i(t)N_i(t)\dot{Q}_i(t)dt = n_i(T_f)N_i(T_f)b_i - \int_0^{T_f} r_i(t)Q_i(t)dt$$

$$i = m+1,\ldots,n \qquad (4.4.19)$$

with

$$r_i(t) = \dot{n}_i(t)N_i(t) + n_i(t)\dot{N}_i(t) \qquad (4.4.20)$$

We can write the cost functional (4.4.17) as:

$$\begin{aligned}
J_o[P_{s_i}(t),P_{h_i}(t),Q_i(t)] = \int_0^{T_f} & [\{(\sum_{i=1}^{m} \alpha_i) + \lambda(t)P_D(t) + \lambda(t)K_{Lo} \\
& + \sum_{i=m+1}^{n} [n_i(T_f)N_i(T_f)b_i - \tfrac{1}{2}n_i(T_f)b_i^2]\} \\
& + \{\sum_{i=1}^{m} [\beta_i -(1-B_{io})\lambda(t)]P_{s_i}(t) \\
& + \sum_{i=m+1}^{n} -[n_i(t) + (1-B_{io})\lambda(t)]P_{h_i}(t) \\
& + \sum_{i=m+1}^{n} -r_i(t)Q_i(t) + \sum_{i=1}^{m} \gamma_i P_{s_i}^2(t) \\
& + \sum_{i=1}^{n}\sum_{j=1}^{n} P_i(t)\lambda(t)B_{ij}P_j(t) \\
& + \sum_{i=m+1}^{n} \frac{E_i}{2}\dot{n}_i(t)Q_i^2(t)\}]dt
\end{aligned} \qquad (4.4.21)$$

Dropping terms in (4.4.21) that are explicitly independent of the control functions $P_{s_i}(t)$, $P_{h_i}(t)$ and $Q_i(t)$ over the interval $[0,T_f]$ it is only necessary to consider minimizing:

$$J_1[P_{s_i}(t),P_{h_i}(t),Q_i(t)] = \int_0^{T_f} [\sum_{i=1}^{m} [\beta_i - (1-B_{io})\lambda(t)]$$

$$P_{s_i}(t) + \sum_{i=m+1}^{n} -[n_i(t) +(1-B_{io})\lambda(t)]$$

$$P_{h_i}(t) + \sum_{i=m+1}^{n} -r_i(t)Q_i(t)$$

$$+ \sum_{i=1}^{m} \gamma_i P_{s_i}{}^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} P_i(t)\lambda(t)B_{ij}P_j(t)$$

$$+ \sum_{i=m+1}^{n} \frac{E_i}{2} \dot{n}_i(t)Q_i{}^2(t)]dt$$

(4.4.22)

Define the (2n-m)x1 column vectors.

$$\underline{u}(t) = \text{col.}[\underline{P}_s(t),\underline{P}_h(t),\underline{Q}(t)] \qquad (4.4.23)$$

$$\underline{L}(t) = \text{col.}[\underline{L}_s(t),\underline{L}_h(t),\underline{L}_Q(t)] \qquad (4.4.24)$$

where

$$\underline{P}_s(t) = \text{col.}[P_{s_1}(t),\ldots,P_{s_m}(t)] \qquad (4.4.25)$$

$$\underline{P}_h(t) = \text{col.}[P_{h_{m+1}}(t),\ldots,P_{h_n}(t)] \qquad (4.4.26)$$

$$\underline{L}_s(t) = \text{col.}[\{\beta_1 -(1-B_{io})\lambda(t)\},\ldots,\{\beta_m -(1-B_{mo})\lambda(t)]$$

(4.4.27)

$$\underline{L}_h(t) = \text{col.}[\{-[n_{m+1}(t) + (1-B_{(m+1)o})\lambda(t)]\},\ldots,$$

$$\{-[n_n(t) + (1-B_{no})\lambda(t)]\}] \qquad (4.4.28)$$

$$\underline{L}_Q(t) = \text{col.}[-r_{m+1}(t),\ldots,-r_n(t)] \tag{4.4.29}$$

the (2n-m)x(2n-m) matrix $\underline{B}(t)$:

$$\underline{B}(t) = \begin{bmatrix} \underline{B}_s(t) & \underline{B}_{sh}(t) & \underline{0} \\ \underline{B}_{hs}(t) & \underline{B}_h(t) & \underline{0} \\ \underline{0} & \underline{0} & \underline{B}_Q(t) \end{bmatrix} \tag{4.4.30}$$

where

$$\underline{B}_s(t) = (b_{ij_s}(t)) \tag{4.4.31}$$

is the (mxm) matrix whose elements are:

$$b_{ii_s}(t) = \gamma_i + \lambda(t)B_{ii} \qquad i = 1,\ldots,m \tag{4.4.32}$$

$$b_{ij_s}(t) = \lambda(t)B_{ij} \qquad i \neq j \quad i,j = 1,\ldots,m \tag{4.4.33}$$

$$\underline{B}_{sh}(t) = (b_{ij_{sh}}(t)) \tag{4.4.34}$$

is the mx(n-m) matrix whose elements are:

$$b_{ij_{sh}}(t) = \lambda(t)B_{ij} \qquad \begin{array}{l} i = 1,\ldots,m \\ j = m+1,\ldots,n \end{array}$$

$$\underline{B}_{hs}(t) = (b_{ij_{hs}}(t)) \tag{4.4.35}$$

is the (n-m)xm matrix whose elements are:

$$b_{ij_{hs}}(t) = \lambda(t)B_{ij} \qquad \begin{array}{l} i = m+1,\ldots,n \\ j = 1,\ldots,m \end{array} \tag{4.4.36}$$

and $\underline{B}_Q(t)$ is the (n-m)x(n-m) diagonal matrix:

$$\underline{B}_Q(t) = \text{diag}[\frac{E_{m+1}\dot{n}_{m+1}(t)}{2}, \ldots, \frac{E_n\dot{n}_n(t)}{2}] \tag{4.4.37}$$

Using these definitions (4.4.22) becomes:

$$J_1(\underline{u}(t)) = \int_0^{T_f} \{\underline{L}^T(t)\underline{u}(t) + \underline{u}^T(t)\underline{B}(t)\underline{u}(t)\}dt \tag{4.4.38}$$

Let

$$\underline{V}^T(t) = \underline{L}^T(t)\underline{B}^{-1}(t) \tag{4.4.39}$$

and we have

$$\underline{B}^{-1}(t) = \begin{bmatrix} \underline{C}_s(t) & \underline{C}_{sh}(t) & \underline{0} \\ \underline{C}_{hs}(t) & \underline{C}_h(t) & \underline{0} \\ \underline{0} & \underline{0} & \underline{C}_Q(t) \end{bmatrix} \tag{4.4.40}$$

where

$$\underline{C}_s(t) = [\underline{B}_s(t) - \underline{B}_{sh}(t)\underline{B}_h^{-1}(t)\underline{B}_{hs}(t)]^{-1} \tag{4.4.41}$$

$$\underline{C}_h(t) = [\underline{B}_h(t) - \underline{B}_{hs}(t)\underline{B}_s^{-1}(t)\underline{B}_{sh}(t)]^{-1} \tag{4.4.42}$$

$$\underline{C}_{sh}(t) = [-\underline{B}_s^{-1}(t)\underline{B}_{sh}(t)\underline{C}_h(t)] \tag{4.4.43}$$

$$\underline{C}_{hs}(t) = [-\underline{B}_h^{-1}(t)\underline{B}_{hs}(t)\underline{C}_s(t)] \tag{4.4.44}$$

$$\underline{C}_Q(t) = \underline{B}_Q^{-1}(t) \tag{4.4.45}$$

provided that the inverses in the last equalities exist.

(4.4.39) can then be expressed as:

$$\underline{V}^T(t) = [\underline{V}_s^T(t)|\underline{V}_h^T(t)|\underline{V}_Q^T(t)] \tag{4.4.46}$$

where

$$\underline{V}_s^T(t) = \underline{L}_s^T(t)\underline{C}_s(t) + \underline{L}_h^T(t)\underline{C}_{hs}(t) \tag{4.4.47}$$

$$\underline{V}_h^T(t) = \underline{L}_s^T(t)\underline{C}_{sh}(t) + \underline{L}_h^T(t)\underline{C}_h(t) \tag{4.4.48}$$

$$\underline{V}_Q^T(t) = \underline{L}_Q^T(t)\underline{C}_Q(t) \tag{4.4.49}$$

Furthermore, (4.4.38) can be written as:

$$J_1(\underline{u}(t)) = \int_0^{T_f} \{(\underline{u}(t) + \frac{\underline{V}(t)}{2})^T \underline{B}(t)(\underline{u}(t) + \frac{\underline{V}(t)}{2})]$$

$$- (\frac{\underline{V}^T(t)}{2}\, \underline{B}(t)\, \frac{\underline{V}(t)}{2})\}dt \qquad (4.4.50)$$

The last term in the integrand of (4.4.50) does not depend explicitly on $\underline{u}(t)$, so that one needs to consider only minimizing:

$$J_2(\underline{u}(t)) = \int_0^{T_f} \{[\underline{u}(t) + \frac{\underline{V}(t)}{2}]^T \underline{B}(t)[\underline{u}(t) + \frac{\underline{V}(t)}{2}]\}dt \qquad (4.4.51)$$

subject to the constraints.

$$Q_i(T_f) = b_i \qquad i = m+1,\ldots,n \qquad (4.4.52)$$

Define the (n-m)x1 column vector:

$$\underline{b} = \text{col.}[b_{m+1},\ldots,b_n] \qquad (4.4.53)$$

and the (2n-m)x(n-m) matrix $\underline{M}$ by:

$$\underline{M}^T = [\underline{0}|\underline{I}\ \frac{d}{dt}] \qquad (4.4.54)$$

$\underline{0}$ being the (n-m)xn matrix whose elements are all zero, and I being the (n-m)x(n-m) unity matrix.

Thus the constraints of equation (4.4.53) reduce to:

$$\underline{b} = \int_0^{T_f} \underline{M}^T \underline{u}(t)dt \qquad (4.4.55)$$

The control vector $\underline{u}(t)$ is considered an element of the Hilbert Space $L_{2,\underline{B}}^{(2n-m)}[0,T_f]$ of the (2n-m) vector valued square integrable functions defined on $[0,T_f]$ whose inner product is given by

$$<\underline{V}(t),\underline{u}(t)> = \int_0^{T_f} \underline{V}^T(t)\underline{B}(t)\underline{u}(t)dt \qquad (4.4.56)$$

for every $\underline{V}(t)$ and $\underline{u}(t)$ in $L_{2,B}^{(2n-m)}[0,T_f]$, provided that $\underline{B}(t)$ is positive definite.

The given vector $\underline{b}$ is considered an element of the real space $R^{(n-m)}$ with the Euclidean inner product definition:

$$<\underline{X},\underline{Y}> = \underline{X}^T\underline{Y} \qquad (4.4.57)$$

for every $\underline{X}$ and $\underline{Y}$ in $R^{(n-m)}$.

Equation (4.4.55) then defines a bounded linear transformation $T: L_{2,B}^{(2n-m)}[0,T_f] \rightarrow R^{(n-m)}$. This can be written as:

$$\underline{b} = T[\underline{u}(t)] \qquad (4.4.58)$$

and the cost functional given in (4.4.51) reduces to

$$J_2(\underline{u}(t)) = ||\underline{u}(t) + \frac{\underline{V}(t)}{2}||^2 \qquad (4.4.59)$$

Finally, it is only necessary to minimize

$$J(\underline{u}(t)) = ||\underline{u}(t) + \frac{\underline{V}(t)}{2}|| \qquad (4.4.60)$$

subject to

$$\underline{b} = T[\underline{u}(t)]$$

for a given $\underline{b}$ in $R^{(n-m)}$.

4.4.3 The Optimal Solution

The optimal solution to the problem formulated in the previous subsection using the results of Chapter 2 is given by:

$$\underline{u}_\xi(t) = T^\dagger[\underline{b} + T(\underline{V}/2)] - (\underline{V}/2) \qquad (4.4.61)$$

where $T^\dagger$ is obtained as follows:

T^*, the adjoint of T, is obtained using the identity:

$$<\underline{\xi},\underline{Tu}>_{R^{n-m}} = <\underline{T^*\xi}, \underline{u}>_{L_{2,B}^{(2n-m)}[0,T_f]} \qquad (4.4.62)$$

Let

$$\underline{\xi} = \text{col.}[\xi_{m+1},\ldots,\xi_n] \qquad (4.4.63)$$

$$\underline{T^*\xi} = \text{col.}[\underline{T}_p, \underline{T}_h, \underline{T}_Q] \qquad (4.4.64)$$

with

$$\underline{T}_p = \text{col.}[t_1, \ldots, t_m] \tag{4.4.65}$$

$$\underline{T}_h = \text{col.}[t_{m+1}, \ldots, t_n] \tag{4.4.66}$$

$$\underline{T}_Q = \text{col.}[\omega_{m+1}, \ldots, \omega_n] \tag{4.4.67}$$

then in $R^{(n-m)}$, the inner product of the left hand side of (4.4.62) is:

$$<\underline{\xi},\underline{Tu}> = \underline{\xi}^T \int_0^{T_f} \underline{M}^T \underline{u}(t)dt \tag{4.4.69}$$

where $Q_i(o) = 0$ according to (4.4.13).

And in $L_{2,B}^{(2n-m)}[0,T_f]$, the inner product of the right hand side of (4.4.62) is

$$<\underline{T^*\xi}, \underline{u}> = \int_0^{T_f} (\underline{T^*\xi})^T \underline{B}(t)\, \underline{u}(t)dt \tag{4.4.70}$$

Using (4.4.64), (4.4.30) and (4.4.23) this reduces to:

$$\begin{aligned} <\underline{T^*\xi},\underline{u}> = \int_0^{T_f} \{&[\underline{T}_p{}^T \underline{B}_s + \underline{T}_h{}^T \underline{B}_{hs}]\underline{P}_s \\ &+ [\underline{T}_p{}^T \underline{B}_{sh} + \underline{T}_h{}^T \underline{B}_h]\underline{P}_h \\ &+ \underline{T}_Q{}^T \underline{B}_Q \underline{Q}\}dt \end{aligned} \tag{4.4.71}$$

Thus the identity of (4.4.62) using (4.4.69) and (4.4.71) reduces to:

$$\begin{aligned} \underline{\xi}^T \underline{Q}(T_f) = \int_0^{T_f} \{&[\underline{T}_p{}^T \underline{B}_s + \underline{T}_h{}^T \underline{B}_{hs}]\underline{P}_s(t) \\ &+ [\underline{T}_p{}^T \underline{B}_{sh} + \underline{T}_h{}^T \underline{B}_h]\underline{P}_h(t) \\ &+ \underline{T}_Q{}^T \underline{B}_Q \underline{Q}(t)\}dt \end{aligned} \tag{4.4.72}$$

Equation (4.4.72) is satisfied for the choice:

$$\underline{T}_p(t) = \underline{0} \qquad t\,\varepsilon[0,T_f] \tag{4.4.73}$$

$$\underline{T}_h(t) = 0 \qquad t\,\varepsilon[0,T_f] \tag{4.4.74}$$

$$\underline{T}_Q(t) = \underline{0} \qquad t\,\varepsilon[0,T_f) \tag{4.4.75}$$

$$\underline{T}_Q{}^T(T_f) = \underline{\xi}^T \underline{B}_Q^{-1}(T_f) \tag{4.4.76}$$

Thus $\underline{T^*\xi}$ is given by:

$$T^*[\underline{\xi}] = \left[\frac{\underline{0}}{\underline{\mu}(t)}\right] \underline{\xi} \tag{4.4.77}$$

where $\underline{0}$ is the nx(n-m) matrix whose elements are all zeros, and $\underline{u}(t)$ is the (n-m)x(n-m) diagonal matrix given by:

$$\underline{\mu}(t) = \mathrm{diag}[\theta_{m+1}(t),\ldots,\theta_n(t)] \tag{4.4.78}$$

$$\theta_i(t) = 0 \qquad t\,\varepsilon[0,T_f)$$

$$= \frac{2}{E_i\ \hbar_i(T_f)} \qquad t = T_f \tag{4.4.79}$$

The operator J is next evaluated as:

$$J(\underline{\xi}) = T[\underline{T^*\xi}] \tag{4.4.80}$$

Using (4.4.54), (4.4.55) and (4.4.77) this reduces to:

$$J(\underline{\xi}) = \int_0^{T_f} [\underline{0}|\underline{I}\ \frac{d}{dt}]\left[\frac{\underline{0}}{\underline{\mu}(t)}\right] \underline{\xi} dt \tag{4.4.81}$$

or

$$J(\xi) = \int_0^{T_f} \frac{d}{dt}\ \underline{\mu}(t)\ \underline{\xi}\ dt \tag{4.4.82}$$

so that

$$J(\xi) = [\underline{\mu}(T_f) - \underline{\mu}(o)]\underline{\xi} \tag{4.4.83}$$

but by (4.4.79) $\underline{\mu}(o) = 0$, then

$$J(\underline{\xi}) = \underline{\mu}(T_f)\underline{\xi} \tag{4.4.84}$$

This yields:

$$J^{-1}(\xi) = \underline{\mu}^{-1}(T_f)\underline{\xi} \tag{4.4.85}$$

Finally, the pseudo-inverse operator $T^{\dagger}$ is obtained from the definition

$$T^{\dagger}[\xi] = T^{*}[J^{-1}\underline{\xi}] \tag{4.4.86}$$

using (4.4.77) and (4.4.85) this yields:

$$T^{\dagger}\underline{\xi} = \left[\frac{0}{\underline{\mu}(t)}\right] \underline{\mu}^{-1}(T_f)\underline{\xi} \tag{4.4.87}$$

or

$$T^{\dagger}\underline{\xi} = \left[\frac{0}{\mu(t)\mu^{-1}(T_f)}\right]\underline{\xi} \tag{4.4.88}$$

From (4.4.54) and (4.4.46) one obtains

$$T(\underline{V}(t)/2) = \frac{1}{2}[\underline{V}_Q(T_f) - \underline{V}_Q(o)] \tag{4.4.89}$$

The optimal solution is now found by substituting (4.4.89), (4.4.88) and (4.4.46) in (4.4.62);

$$\underline{P}_{s_\xi}(t) = -\frac{1}{2}\underline{V}_s(t) \tag{4.4.90}$$

$$\underline{P}_{h_\xi}(t) = -\frac{1}{2}\underline{V}_h(t) \tag{4.4.91}$$

$$\underline{Q}_\xi(t) = \underline{\mu}(t)\mu^{-1}(T_f)[\underline{b} + \frac{1}{2}\{\underline{V}_Q(T_f) - \underline{V}_Q(o)\}] - \frac{\underline{V}_Q(t)}{2} \tag{4.4.92}$$

which can be reduced by using (4.4.79):

$$\underline{Q}_\xi(t) = -\frac{\underline{V}_Q(t)}{2} \qquad t\ \varepsilon[o,T_f] \tag{4.4.93}$$

$$Q_\xi(T_f) = \underline{b} \tag{4.4.94}$$

since $\underline{Q}_{\xi}(o) = -\frac{\underline{V}_Q(o)}{2} = \underline{0}$.

Thus according to (4.4.57), (4.4.48) and (4.4.49) the optimal solution is:

$$\underline{P}_{s_\xi}(t) = -\frac{1}{2}[\underline{C}_s^T(t)\underline{L}_s(t) + \underline{C}_{hs}^T(t)\underline{L}_h(t)] \tag{4.4.95}$$

$$\underline{P}_{h_\xi}(t) = -\frac{1}{2}[\underline{C}_{sh}^T(t)\underline{L}_s(t) + \underline{C}_h^T(t)\underline{L}_h(t)] \tag{4.4.96}$$

$$\underline{Q}_\xi^T(t) = -\frac{1}{2}[\underline{C}_Q^T(t)\underline{L}_Q(t)] \tag{4.4.97}$$

It is noted that the optimal solution involves the unknown functions $\lambda(t)$ and $n_i(t)$ which are to be determined such that the constraints (4.4.4) and (4.4.15) are satisfied.

4.4.4. Comparison with Kron's [11] Equations

Consider an electric power system with one thermal and one hydro plant. The optimal solution for this system is given by (4.4.95), (4.4.96) and (4.4.97) where all vector quantities reduce to one element vectors, since $m = 1$ and $n\text{-}m = 1$ in this case.

Here we have using (4.4.27), (4.4.28) and (4.4.29)

$$L_s(t) = \{\beta_1 - \lambda(t)[1 - B_{10}]\} \tag{4.4.98}$$

$$L_h(t) = -\{n_2(t) + \lambda(t)[1 - B_{20}]\} \tag{4.4.99}$$

$$L_Q(t) = -r_2(t) \tag{4.4.100}$$

And using (4.4.41), (4.4.42), (4.4.43), (4.4.44) and (4.4.45)

$$C_s(t) = \frac{B_{22}}{(\gamma_1 + \lambda(t)B_{11})B_{22} - \lambda(t)B_{12}B_{21}} \tag{4.4.101}$$

$$C_h(t) = \frac{(\gamma_1 + \lambda(t)B_{11})}{\lambda(t)[(\gamma_1 + \lambda(t)B_{11})B_{22} - \lambda(t)B_{12}B_{21}]} \tag{4.4.102}$$

$$C_{sh}(t) = -\frac{B_{12}\lambda(t)}{\gamma_1 + \lambda(t)B_{11}} C_h(t) \tag{4.4.103}$$

$$C_{hs}(t) = -\frac{B_{21}}{B_{22}} C_s(t) \tag{4.4.104}$$

$$C_Q(t) = \frac{2}{E_2\hat{n}_2(t)} \tag{4.4.105}$$

Thus the optimal solution is:

$$P_{s_\xi}(t) = -\frac{B_{22}}{2[(\gamma_1 + \lambda(t)B_{11})B_{22} - \lambda(t)B_{12}B_{21}]}$$

$$[\{\beta_1 - \lambda(t)[1 - B_{10}]\} + \frac{B_{21}}{B_{22}}\{n_2(t)$$

$$+ \lambda(t)[1 - B_{20}]\}] \tag{4.4.106}$$

$$P_{h_\xi}(t) = -\frac{[\gamma_1 + \lambda(t)B_{11}]}{2\lambda(t)[(\gamma_1 + \lambda(t)B_{11})B_{22} - \lambda(t)B_{12}B_{21}]}$$

$$\left[-\frac{B_{12}\lambda(t)}{\gamma_1 + \lambda(t)B_{11}}\{\beta_1 - \lambda(t)[1 - B_{10}]\}\right.$$

$$\left.- \{n_2(t) + \lambda(t)[1 - B_{20}]\}\right] \tag{4.4.107}$$

$$Q_\xi(t) = \frac{n_2(t)}{E_2\hat{n}_2(t)} \tag{4.4.108}$$

Furthermore, eliminating $n_2(t)$ between (4.4.106) and (4.4.107) yields

$$\frac{2\Delta}{B_{22}} P_{s_\xi}(t) + [\beta_1 - \lambda(t)(1 - B_{10})] + \frac{B_{21}}{B_{22}}[\frac{2\lambda(t)\Delta}{\gamma_1 + \lambda(t)B_{11}}$$

$$P_{h_\xi}(t) - \frac{B_{12}\lambda(t)}{\gamma_1 + \lambda(t)B_{11}}[\beta_1 - \lambda(t)(1 - B_{10})]] = 0 \tag{4.4.109}$$

where

$$\Delta = (\gamma_1 + \lambda(t)B_{11})B_{22} - \lambda(t)B_{12}B_{21} \tag{4.4.110}$$

Equation (4.4.109) can be rewritten as:

$$\frac{2\Delta}{B_{22}} P_{s_\xi}(t) + [\beta_1 - \lambda(t)(1 - B_{10})]\frac{\Delta}{B_{22}(\gamma_1 + \lambda(t)B_{11})}$$

$$+ \frac{2\Delta\ \lambda(t)B_{21}}{B_{22}(\gamma_1 + \lambda(t)B_{11})} P_{h_\xi}(t) = 0$$

or

$$2P_{s_\xi}(t)[\gamma_1 + \lambda(t)B_{11}] + [\beta_1 - \lambda(t)(1 - B_{10})]$$

$$+ 2B_{21}\lambda(t)P_{h_\xi}(t) = 0 \tag{4.4.111}$$

This equation should be satisfied if $P_{s_\xi}(t)$ and $P_{h_\xi}(t)$ are to be optimal.

The optimal volume of water discharge is reduced to:

$$Q_\xi(t) = \frac{1}{E_2}[N_2(t) + \dot{N}_2(t)\frac{n_2(t)}{\dot{n}_2(t)}] \tag{4.4.112}$$

This results from applying (4.4.20).

Furthermore (4.4.106) and (4.4.107) can be written as:

$$2\Delta P_{s_\xi}(t) = - B_{22}[\beta_1 - \lambda(t)(1 - B_{10}) - B_{21}[n_2(t) + \lambda(t)(1 - B_{20})] \tag{4.4.113}$$

$$2\Delta P_{h_\xi}(t) = B_{12}[\beta_1 - \lambda(t)(1 - B_{10})] + \frac{(\gamma_1 + \lambda(t)B_{11})}{\lambda(t)}$$

$$[n_2(t) + \lambda(t)(1 - B_{20})] \tag{4.4.114}$$

Multiplying both sides of (4.4.113) by B_{12}, (4.4.114) by B_{22} and adding, the following is obtained.

$$2[B_{12}P_{s_\xi}(t) + B_{22}P_{h_\xi}(t)] = \frac{n_2(t) + \lambda(t)(1 - B_{20})}{\lambda(t)}$$

or

$$n_2(t) = \lambda(t)[2B_{12}P_{s_\xi}(t) + 2B_{22}P_{h_\xi}(t) - (1 - B_{20})] \tag{4.4.115}$$

If one lets

$$D_\xi(t)S_\xi(t) = (1 - B_{20}) - 2B_{12}P_{s_\xi}(t) - 2B_{22}P_{h_\xi}(t) \tag{4.4.116}$$

then
$$n_2(t) = -\lambda(t)D_\xi(t) \tag{4.4.117}$$

$$\dot{n}_2(t) = -[\dot{\lambda}(t)D_\xi(t) + \lambda(t)\dot{D}_\xi(t)] \tag{4.4.118}$$

so that the optimal volume of water discharged given by (4.4.112) becomes:

$$Q_\xi(t) = \frac{1}{E_2}[N_2(t) + \dot{N}_2(t) \frac{\lambda(t)D_\xi(t)}{\dot{\lambda}(t)D_\xi(t) + \lambda(t)\dot{D}_\xi(t)} \tag{4.4.119}$$

Kron's scheduling equations [11] are:

$$\frac{\partial F}{\partial P_s} + \lambda(t) \frac{\partial P_L}{\partial P_s} - \lambda(t) = 0 \tag{4.4.120}$$

$$\frac{\lambda(t)}{S} [1 - \frac{\partial P_L}{\partial P_h}] \frac{\partial P_h}{\partial h} + \frac{d}{dt} [\lambda(t)(1 - \frac{\partial P_L}{\partial P_h}) \frac{\partial P_h}{\partial \dot{Q}}] = 0 \tag{4.4.121}$$

In the particular power system under consideration one obtains

$$\frac{\partial F}{\partial P_s} = \beta_1 + 2\gamma_1 P_s(t) \tag{4.4.122}$$

$$\frac{\partial P_L}{\partial P_s} = 2B_{11}P_s(t) + 2B_{21}P_h(t) + B_{10} \tag{4.4.123}$$

Thus (4.4.120) becomes:

$$2P_s(t)[\gamma_1 + \lambda(t)B_{11}] + [\beta_1 - \lambda(t)(1 - B_{10})] + 2B_{21}\lambda(t) \cdot$$

$$P_h(t) = 0 \tag{4.4.124}$$

This is exactly (4.4.111), thus the optimal solution obtained in subsection (4.4.3) is Kron's first equation.

Furthermore, one finds that

$$1 - \frac{\partial P_L}{\partial P_h} = 1 - B_{20} - 2B_{22}P_h(t) - 2B_{12}P_s(t)$$

And by (4.4.116) this reduces to:

$$D(t) = 1 - \frac{\partial P_L}{\partial P_h} \tag{4.4.125}$$

Thus Kron's second equation (4.4.121) transforms to:

$$\frac{\lambda(t)D(t)}{s_2} \frac{\partial P_h}{\partial h} + \frac{d}{dt}[\lambda(t)D(t) \frac{\partial P_h}{\partial \dot{Q}}] = 0 \tag{4.4.126}$$

By (4.4.7), (4.4.14) one obtains

$$\frac{\partial P_h}{\partial h} = \frac{\dot{Q}_2(t)}{G_2} \tag{4.4.127}$$

And by (4.4.15)

$$\frac{\partial P_h}{\partial \dot{Q}} = N_2(t) - E_2Q_2(t) \tag{4.4.128}$$

Using (4.4.127), (4.4.128) and (4.4.11), equation (4.4.125) becomes:

$$E_2\lambda(t)D_2(t)\dot{Q}_2(t) + \frac{d}{dt}[\lambda(t)D_2(t)[N_2(t) - E_2Q_2(t)]] = 0$$

or

$$\lambda(t)\frac{d}{dt} [D_2(t)N_2(t)] + \dot{\lambda}(t)D_2(t)N_2(t) - E_2Q_2(t)[\dot{\lambda}(t)D_2(t) + \lambda(t)\dot{D}_2(t)] = 0 \tag{4.4.129}$$

This yields

$$Q_2(t) = \frac{1}{E_2}[N_2(t) + \frac{\dot{N}_2(t)\lambda(t)D_2(t)}{\dot{\lambda}(t)D_2(t) + \lambda(t)\dot{D}_2(t)}] \tag{4.4.130}$$

But this is precisely (4.4.119), so that the optimal solution obtained is the second equation given by Kron.

4.4.5 Implementing the Optimal Solution

The optimal solution obtained in subsection (4.4.3) contains the unknown functions $n_i(t)$ and $\lambda(t)$. These can be determined by substituting the optimal solution in the corresponding constraint equations. The resulting equations in $n_i(t)$ and $\lambda(t)$ are generally nonlinear. In order to get deeper insight into how the actual solution is obtained, the following simplifying assumptions are made:

1. The system is characterized by a loss formula where:

   a. $B_{ij} = 0 \qquad i \neq j \qquad i = 1,\ldots,n$

   b. $B_{io} = 0 \qquad\qquad i = 1,\ldots,n$

   c. $K_{Lo} = 0$

2. The rate of natural water inflow to the reservoirs is constant.

These assumptions will represent no loss of generality.

The first assumption implies the following in (4.4.30).

$$\underline{B}_{sh}(t) = 0 \tag{4.4.131}$$

$$\underline{B}_{hs}(t) = 0 \tag{4.4.132}$$

$$\underline{B}_{s}(t) = \text{diag}[(\gamma_1 + \lambda(t)B_{11}),\ldots,(\gamma_m + \lambda(t)B_{mm})] \tag{4.4.133}$$

$$\underline{B}_{h}(t) = \text{diag}[\lambda(t)B_{(m+1)(m+1)},\ldots,\lambda(t)B_{nn}] \tag{4.4.134}$$

And in (4.4.27) and (4.4.28):

$$\underline{L}_{s}(t) = \text{col.}[\{\beta_1 - \lambda(t)\},\ldots,\{\beta_m - \lambda(t)\}] \tag{4.4.135}$$

$$\underline{L}_{h}(t) = \text{col.}[\{-[n_{m+1}(t) + \lambda(t)]\},\ldots,\{-[n_n(t) + \lambda(t)]\}] \tag{4.4.136}$$

Using (4.4.131), (4.4.132),(4.4.133) and (4.4.134) in (4.4.41) through (4.4.44) one obtains:

$$\underline{C}_s(t) = \text{diag}\left[\frac{1}{\gamma_1 + \lambda(t)B_{11}}, \ldots, \frac{1}{\gamma_m + \lambda(t)B_{mm}}\right] \qquad (4.4.137)$$

$$\underline{C}_h(t) = \text{diag}\left[\frac{1}{\lambda(t)B_{(m+1)(m+1)}}, \ldots, \frac{1}{\lambda(t)B_{nn}}\right] \qquad (4.4.138)$$

$$\underline{C}_{hs}(t) = \underline{0} \qquad (4.4.139)$$

$$\underline{C}_{sh}(t) = \underline{0} \qquad (4.4.140)$$

Thus the optimal solution given by (4.4.95), (4.4.96) and (4.4.97) reduces to:

$$\underline{P}_{s_\xi}(t) = -\frac{1}{2}\,\text{col.}\left[\frac{\beta_1 - \lambda(t)}{\gamma_1 + \lambda(t)B_{11}}, \ldots, \frac{\beta_m - \lambda(t)}{\gamma_m + \lambda(t)B_{mm}}\right] \qquad (4.4.141)$$

$$\underline{P}_{h_\xi}(t) = -\frac{1}{2}\,\text{col.}\left[-\frac{[n_{m+1}(t) + \lambda(t)]}{\lambda(t)B_{(m+1)(m+1)}}, \ldots, -\frac{[n_n(t) + \lambda(t)]}{\lambda(t)B_{nn}}\right] \qquad (4.4.142)$$

$$\underline{Q}_\xi(t) = \text{col.}\left[\left\{\frac{1}{E_{m+1}}\left(N_{m+1}(t) + \dot{N}_{m+1}(t)\,\frac{n_{m+1}(t)}{\dot{n}_{m+1}(t)}\right)\right\}, \ldots, \left\{\frac{1}{E_n}\left(N_n(t) + \dot{N}_n(t)\,\frac{n_n(t)}{\dot{n}_n(t)}\right)\right\}\right] \qquad (4.4.143)$$

Or component wise this is given by:

$$P_{s_{i_\xi}}(t) = \frac{\lambda(t) - \beta_i}{2[\gamma_i + \lambda(t)B_{ii}]} \qquad i = 1,\ldots,m \qquad (4.4.144)$$

$$P_{h_{i_\xi}}(t) = \frac{n_i(t) + \lambda(t)}{2\lambda(t)B_{ii}} \qquad i = m+1,\ldots,n \qquad (4.4.145)$$

$$Q_{i_\xi}(t) = \frac{1}{E_i}\left[N_i(t) + \dot{N}_i(t)\,\frac{n_i(t)}{\dot{n}_i(t)}\right] \qquad i = m+1,\ldots,n \qquad (4.4.146)$$

The constraint equation (4.4.4) for the optimal solution becomes:

$$P_D(t) = \sum_{i=1}^{m} P_{s_{i_\xi}}(t) + \sum_{i=m+1}^{n} P_{h_{i_\xi}}(t)$$

$$- \sum_{i=1}^{m} B_{ii} P_{s_{i_\xi}}^{2}(t) - \sum_{i=m+1}^{n} B_{ii} P_{h_{i_\xi}}^{2}(t) \qquad (4.4.147)$$

Substituting (4.4.144) and (4.4.145) this reduces to:

$$P_D(t) = \beta - \sum_{i=1}^{m} C_i \frac{1}{[\gamma_i + B_{ii}\lambda(t)]^2} - \sum_{i=m+1}^{n} D_i \left[\frac{n_i(t)}{\lambda(t)}\right]^2 \qquad (4.4.148)$$

where

$$\beta = \sum_{i=1}^{n} \frac{1}{4B_{ii}} \qquad (4.4.149)$$

$$C_i = \frac{[\beta_i B_{ii} + \gamma_i]^2}{4B_{ii}} \qquad i = 1,\ldots,m \qquad (4.4.150)$$

$$D_i = \frac{1}{4B_{ii}} \qquad i = 1,\ldots,m \qquad (4.4.151)$$

The hydro-power constraint given by (4.4.15) is

$$P_{h_{i_\xi}}(t) = \dot{Q}_{i_\xi}(t)[N_i(t) - E_i Q_{i_\xi}(t)] \qquad i = m+1,\ldots,n \qquad (4.4.152)$$

Let

$$x_i(t) = \frac{n_i(t)}{\dot{n}_i(t)} \qquad i = m+1,\ldots,n \qquad (4.4.153)$$

or

$$n_i(t) = n_i(o)\exp\left[\int_0^t [x_i(t)]^{-1}dt\right] \qquad i = m+1,\ldots,n \qquad (4.4.154)$$

Then (4.4.146) becomes:

$$Q_{i_\xi}(t) = \frac{1}{E_i}[N_i(t) + \dot{N}_i(t)x_i(t)] \qquad i = m+1,\ldots,n \qquad (4.4.155)$$

Differentiating both sides of (4.4.155) yields:

$$\dot{Q}_{i_\xi}(t) = \frac{1}{E_i}[\dot{N}_i(t) + \ddot{N}_i(t)x_i(t) + \dot{N}_i(t)\dot{x}_i(t)]$$

$$i = m+1,\ldots,n \qquad (4.4.156)$$

so that the constraints (4.4.152) reduce to

$$P_{h_{i_\xi}}(t) = -\frac{\dot{N}_i(t)x_i(t)}{E_i}[\dot{N}_i(t) + \ddot{N}_i(t)x_i(t) + \dot{N}_i(t)\dot{x}_i(t)]$$

$$i = m+1,\ldots,n \qquad (4.4.157)$$

Using (4.4.145) in (4.4.157) one obtains:

$$\frac{1}{2B_{ii}} + \frac{n_i(o)\exp[\int_0^t[x_i(t)]^{-1}dt]}{2\lambda(t)B_{ii}} + \frac{\dot{N}_i(t)x_i(t)}{E_i}$$

$$[\dot{N}_i(t) + \ddot{N}_i(t)x_i(t) + \dot{N}_i(t)\dot{x}_i(t)] = 0$$

$$i = m+1,\ldots,n \qquad (4.4.158)$$

Recalling (4.4.10) for $N_i(t)$ and using (4.4.11) one obtains:

$$N_i(t) = \frac{h_i(o)}{G_i} + E_i\int_0^t i_i(\sigma)d\sigma \qquad i = m+1,\ldots,n \qquad (4.4.159)$$

Differentiating one obtains:

$$\dot{N}_i(t) = E_i\ i_i(t) \qquad i = m+1,\ldots,n \qquad (4.4.160)$$

$$\ddot{N}_i(t) = E_i\ \dot{i}_i(t) \qquad i = m+1,\ldots,n \qquad (4.4.161)$$

Thus (4.4.158) reduces to

$$1 + \frac{n_i(o)\exp[\int_0^t(x_i(t)^{-1}dt]}{\lambda(t)} + 2B_{ii}E_i\ i_i(t)x_i(t)$$

$$[i_i(t)\{1 + \dot{x}_i(t)\} + x_i(t)\dot{i}_i(t)] = 0$$

$$i = m+1,\ldots,n \qquad (4.4.162)$$

Let

$$a_i(t) = 2B_{ii}E_i \; i_i{}^2(t) \qquad i = m+1,\ldots,n \qquad (4.4.163)$$

Then (4.4.162) reduces to:

$$\dot{x}_i(t) = -\frac{1 + \dfrac{n_i(o)}{\lambda(t)} \exp[\int_0^t (x_i(t)^{-1} dt]}{a_i(t)x_i(t)} - 1 - \frac{\dot{i}_i(t)}{i_i(t)} x_i(t)$$

$$i = m+1,\ldots,n \qquad (4.4.164)$$

Under the assumption of constant water inflow $\dot{i}_i(t) = 0$, thus (4.4.164) reduces to:

$$\dot{x}_i(t) = - \{[1 + \{[n_i(o)\exp[\int_0^t x_i{}^{-1}(t)dt]/\lambda(t)\}]/a_i(t)x_i(t)\} - 1$$

$$i = m+1,\ldots,n \qquad (4.4.165)$$

The optimal $Q_i(t)$ given by (4.4.146) reduces to

$$Q_{i_\xi}(t) = \frac{1}{E_i}[\frac{h_i(o)}{G_i} + E_i\int_0^t i_i(\sigma)d\sigma + E_i \; i_i(t)x_i(t)]$$

$$i = m+1,\ldots,n \qquad (4.4.166)$$

At t = 0 this reduces to:

$$E_iQ_{i_\xi}(o) = \frac{h_i(o)}{G_i} + E_i \; i_i(o)x_i(o) \qquad i = m+1,\ldots,n$$

$$(4.4.167)$$

But at t = 0

$$Q_{i_\xi}(o) = 0$$

Thus

$$x_i(o) = - \frac{s_ih_i(o)}{i_i(o)} \qquad i = m+1,\ldots,n \qquad (4.4.168)$$

At $t = T_f$, equation (4.4.165) is:

$$E_i Q_{i_\xi}(T_f) = \frac{h_i(o)}{G_i} + E_i I_i(T_f) + E_i\ i_i(T_f) x_i(T_f)$$

$$i = m+1,\ldots,n \qquad (4.4.169)$$

But at $t = T_f$

$$Q_{i_\xi}(T_f) = b_i$$

Thus

$$x_i(T_f) = \frac{b_i}{i_i(T_f)} + x_i(o)\,\frac{i_i(o)}{i_i(T_f)} - \frac{I_i(T_f)}{i_i(T_f)}$$

$$i = m+1,\ldots,n \qquad (4.4.170)$$

where

$$I_i(t) = \int_0^t i_i(\sigma)d\sigma \qquad i = m+1,\ldots,n \qquad (4.4.171)$$

In the case when $i_i(t) = i_i$ = constant, (4.4.168) and (4.4.170) reduce to:

$$x_i(o) = -\frac{s_i h_i(o)}{i_i} \qquad i = m+1,\ldots,n \qquad (4.4.172)$$

$$x_i(T_f) = x_i(o) + [\frac{b_i}{i_i} - T_f] \qquad i = m+1,\ldots,n \qquad (4.4.173)$$

Thus solving (4.4.149) and (4.4.165) subject to the boundary conditions (4.4.172) and (4.4.173) completely defines the optimal schedule. It is noted that (4.4.149) is an algebraic equation in $\lambda(t)$ and the (n-m) unknown functions $n_i(t)$. On the other hand (4.4.165) is a set of (n-m) first order nonlinear differential equations in $x_i(t)$, where $x_i(t)$ and $n_i(t)$ are related by (4.4.153).

In the computerized search for the above mentioned unknown functions, it is highly desirable to characterize the region of search. This is done

by utilizing the physical significance of each variable involved. In addition to this, restrictions on the variables can be obtained so that the functional analytic formulation adopted is a valid one.

Consider the optimal thermal power generation expression of (4.4.144). If the thermal power generated is to satisfy the following practical limitation:

$$P_{Min_i} \leq P_{s_{i_\xi}}(t) \leq P_{Max_i} \qquad i = 1,\dots,m$$

Thus using (4.4.144) one obtains

$$\frac{\beta_i + 2\gamma_i P_{Min_i}}{1 - 2B_{ii}P_{Min_i}} \leq \lambda(t) \leq \frac{\beta_i + 2\gamma_i P_{Max_i}}{1 - 2B_{ii}P_{Max_i}} \quad i = 1,\dots,m \tag{4.4.174}$$

Let

$$\lambda_{Min} = \underset{i = 1,\dots,m}{\mathrm{Max}} \left[\frac{\beta_i + 2\gamma_i P_{Min_i}}{1 - 2B_{ii}P_{Min_i}}\right] \tag{4.4.175}$$

$$\lambda_{Max} = \underset{i=1,\dots,m}{\mathrm{Min}} \left[\frac{\beta_i + 2\gamma_i P_{Max_i}}{1 - 2B_{ii}P_{Max_i}}\right] \tag{4.4.176}$$

Then

$$\lambda_{Min} \leq \lambda(t) \leq \lambda_{Max} \tag{4.4.177}$$

Furthermore an expression for the effective head $h_i(t)$ can be obtained utilizing (4.4.8), (4.4.10) and (4.4.146) as:

$$h_i(t) = - G_i\dot{N}_i(t)x_i(t) \qquad i = m+1,\dots,n \tag{4.4.178}$$

Using (4.4.160) and (4.4.11) this yields

$$S_i h_i(t) = -i_i(t)x_i(t) \qquad i = m+1,\dots,n \tag{4.4.179}$$

Let

$$V_i(t) = S_i h_i(t) \qquad i = m+1,\ldots,n \qquad (4.4.180)$$

hence

$$x_i(t) = -\frac{V_i(t)}{i_i(t)} \qquad i = m+1,\ldots,n \qquad (4.4.181)$$

Here $V_i(t)$ is the volume of water stored in the reservoir. In the case when this volume is restricted between upper and lower bounds given by

$$V_{Min_i} \leq V_i(t) \leq V_{Max_i} \qquad i = m+1,\ldots,n \qquad (4.4.182)$$

This yields

$$-\frac{V_{Max_i}}{i_i(t)} \leq x_i(t) \leq -\frac{V_{Min_i}}{i_i(t)} \qquad i = m+1,\ldots,n \qquad (4.4.183)$$

It is evident that

$$x_i(t) < 0 \qquad i = m+1,\ldots,n \qquad (4.4.184)$$

Furthermore, if the hydro-power is restricted such that:

$$P_{h_{Min_i}} \leq P_{h_{i_\xi}}(t) \leq P_{h_{Max_i}} \qquad i = m+1,\ldots,n \qquad (4.4.185)$$

Then utilizing (4.4.145) the following inequality must be satisfied:

$$\lambda(t)[2B_{ii}P_{h_{Min_i}} - 1] \leq n_i(t) \leq \lambda(t)[2B_{ii}P_{h_{Max_i}} - 1]$$

$$i = m+1,\ldots,n \qquad (4.4.186)$$

If

$$2B_{ii}P_{h_{Max_i}} < 1 \qquad i = m+1,\ldots,n \qquad (4.4.187)$$

then $\qquad n_i(t) < 0 \qquad i = m+1,\ldots,n \qquad (4.4.188)$

This combined with (4.4.185) yields

$$\dot{n}_i(t) > 0 \qquad i = m+1,\ldots,n \quad (4.4.189)$$

The restricted search area in the $(n-\dot{n})$ phase plane is shown in Fig. (4-1). It is noted here that (4.4.177) and (4.4.189) guarantee that the matrix $\underline{B}(t)$ in the inner product definition is positive definite.

It is worth mentioning here that it is assumed in the problem formulation that the inequality constraints are not violated. Thus these constraints are not included in the cost functional to obtain the optimal solution. The optimal solution is then implemented in a way that confines the search area to the regions of the space of the unknown functions where the inequality constraints are not violated. This agrees in principle with the nonlinear programming approach to this type of problems. In the Kuhn-Tucker method, unknown multipliers (called the Kuhn-Tucker Multipliers) are associated with the inequality constraints for inclusion in the cost functional. These multipliers are set to zero as long as the inequality constraints are not violated [49]. Thus an optimum solution is obtained by scanning the whole space of the unknown functions. If the solution obtained violates any inequality constraint, the corresponding variable is set to the nearest value that does not violate the constraint. The main difference between that method and the method adopted here is that the search region in the latter is smaller than that of the former. This reduces the computing time considerably and leads to good estimated values for the unknown functions.

Consider Equation (4.4.165) which can be rewritten as:

$$\dot{x}_i(t) = f_i[x_i(t), \lambda(t), n_i(o)] \qquad i = m+1,\ldots,n \quad (4.4.190)$$

where

$$f_i[x_i(t),\lambda(t),n_i(o)] = -\{[1 + \{[n_i(o)\exp[\int_0^t x_i^{-1}(t)dt]$$

$$/\lambda(t)\}]/a_i(t)x_i(t)\}-1$$

$$i = m+1,\ldots,n \qquad (4.4.191)$$

Let

$$\Delta f_i(s) = f_i[x_i^{(1)}(s),\lambda(s),n_i(o)] - f_i[x_i^{(2)}(s),\lambda(s),n_i(o)]$$

$$i = m+1,\ldots,n \qquad (4.4.192)$$

Then using Equation (4.4.191) this reduces to

$$\Delta f_i(s) = \frac{1}{a_i(s)x_i^{(1)}(s)x_i^{(2)}(s)}[(x_i^{(1)}(s) - x_i^{(2)}(s))$$

$$+ \frac{n_i(o)}{\lambda(s)}(x_i^{(1)}e^{\int_0^s (x_i^{(2)}(\sigma))^{-1}d\sigma}$$

$$- x_i^{(2)}e^{\int_0^s (x_i^{(1)}(\sigma))^{-1}d\sigma})] \qquad i = m+1,\ldots,n$$

$$(4.4.193)$$

The following inequality is true for all s:

$$|\Delta f_i(s)| \leq \frac{1}{|a_i(s)x_i^{(1)}(s)x_i^{(2)}(s)|}\{|x_i^{(1)}(s) - x_i^{(2)}(s)|$$

$$+ |\frac{n_i(0)}{\lambda(s)}|\{|x_i^{(1)}(s) - x_i^{(2)}(s)|+|x_i^{(2)}(s)[1 -$$

$$\exp(\int_0^s\{[x_i^{(1)}(\sigma)]^{-1} - [x_i^{(2)}(\sigma)]^{-1}\}d\sigma)]\}$$

$$i = m+1,\ldots,n$$

$$(4.4.194)$$

Here the fact that $\exp\{\int_0^s [x_i^{(2)}(\sigma)]^{-1} d\sigma\} \leq 1$ was used.

Further use is made of the following inequality:

$$|e^Z - 1| \leq |Z| \quad \text{if Re } Z \leq 0$$

so that

$$|g_i(s)| = |x_i^{(2)}(s)[1 - \exp(\int_0^s \{(x_i^{(1)}(\sigma))^{-1} - (x_i^{(2)}(\sigma))^{-1} \, d\sigma]|$$

$$\leq |x_i^{(2)}(\sigma)| . | \int_0^s \{(x_i^{(1)}(\sigma))^{-1} - (x_i^{(2)}(\sigma))^{-1}\} d\sigma| \tag{4.4.195}$$

This can be reduced to

$$|g_i(s)| \leq \frac{|T_f| \cdot |x_i^{(2)}(s)|}{X_i(s)} \max_s |x_i^{(2)}(s) - x_i^{(1)}(s)| \tag{4.4.196}$$

where

$$X_i(s) = \min_s x_i^{(1)}(s) x_i^{(2)}(s) \qquad i = m+1,\ldots,n \tag{4.4.197}$$

Thus (4.4.194) reduces to

$$|\Delta f_i(s)| \leq M \max_{[0,T_f]} |x_i^{(2)}(s) - x_i^{(1)}(s)| \qquad i = m+1,\ldots,n \tag{4.4.198}$$

where

$$M = \frac{1}{\min[a_i(s) x_i^{(1)}(s) x_i^{(2)}(s)]} \{1 + |\frac{n_i(o)}{\min_s \lambda(s)}| . |1$$

$$+ \{|T_f| \max_s |x_i^{(2)}(s)| / X_i(s)\}|\} \qquad i = m+1,\ldots,n \tag{4.4.199}$$

Furthermore (4.4.198) provides exactly a Lipschitz condition given by

$$\max_{[0,T_f]} |f_i[x_i^{(1)}(s),\ldots] - f_i[x_i^{(2)},\ldots]|$$

$$\leq M.\max_{[0,T_f]} |x_i^{(1)}(s) - x_i^{(2)}(s)| \qquad (4.4.200)$$

This means that Picard's iteration process given by:

$$x_i^{(o)}(t) = x_i(o) \qquad (4.4.201)$$

$$x_i^{(m+1)}(t) = x_i(o) + \int_0^t f(x_i^{(m)}(s),\lambda(s),n_i(o))ds \qquad (4.4.202)$$

is guaranteed to converge to a solution of (4.4.190) for given $n_i(o)$ and $\lambda(s)$.

4.4.6 Practical Application

A computer program was written to solve (4.4.165) and (4.4.148) for $x_i(t)$ ($i = m+1,\ldots,n$) and $\lambda(t)$ to obtain the optimum generation schedule Figure (4-2) shows the flow chart for this program. An initial estimate of the function $\lambda(t)$ is made. This was taken as

$$\lambda^{(o)}(t) = \lambda_{Min} \qquad t \in [0,T_f]$$

where λ_{Min} is given by (4.4.177). For each hydro plant, the initial value of $n_i(t)$ was estimated as:

$$n_i^{(o)}(o) = \lambda_{Min}[2B_{ii}P_{h_{i_{Min}}} - 1] \qquad i = m+1,\ldots,n$$

A solution to (4.4.165) was then obtained for each hydro plant. This was done by utilizing the Picard's algorithm given by (4.4.201) and (4.4.202).

The value of $x_i(T_f)$ obtained here was compared with the boundary condition given in (4.4.172). If the error in this step is large, the estimated value $n_i^{(o)}(o)$ was modified to the correct direction that minimizes the error. When all the $x_i(t)$'s were obtained the corresponding $n_i(t)$'s were evaluated using (4.4.154).

Thus equation (4.4.148) becomes a (2m+2)nd order algebraic equation in $\lambda(t)$ for every t. This is solved for $\lambda(t)$ in the region given by (4.4.171). The $\lambda(t)$ obtained was then taken as the initial estimate instead of λ_{Min} and the process is repeated. If the difference between two successive evaluations of $\lambda(t)$ was less than a prespecified amount the iteration process was stopped. The last step is to evaluate the optimum generation schedules as given by (4.4.144), (4.4.145) and (4.4.146).

This program was applied to a hydro-thermal system with four thermal and three hydro-plants whose particulars are summarized in Tables 4.1 and 4.2. The optimal generation schedules and the assumed power demand curves are shown in Fig. (4-3).

Table 4.1

Thermal Plants' Particulars

Plant No:	1	2	3	4
β	4.4	4.3	4.2	4.25
$\alpha x 10^3$	1.2	1.56	1.67	1.32
$B_{ii} x 10^4$	1.6	1.5	1.8	1.4

Table 4.2

Hydro Plants' Particulars

Plant No:	5	6	7
Constant $i_i(t) x 10^{-6}$	0.1	1.0	0.15
$S_i \times 10^{-10}$ ft^2	7.2	0.72	1.44
$h_i(c)$ ft	100	200	150
$b_i \times 10^{-10}$ ft^3	0.05	0.35	0.78
$B_{ii} \times 10^4$	2.2	2.3	2.4

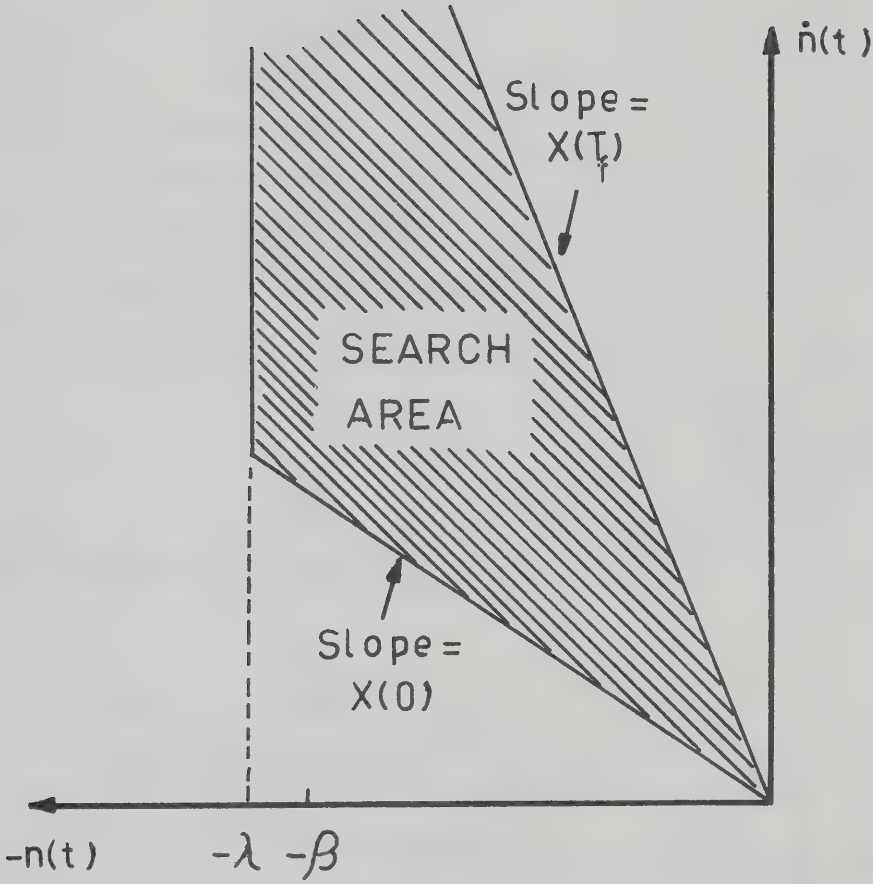


Figure 4.1 The Search Area in the ṅ-n plane.

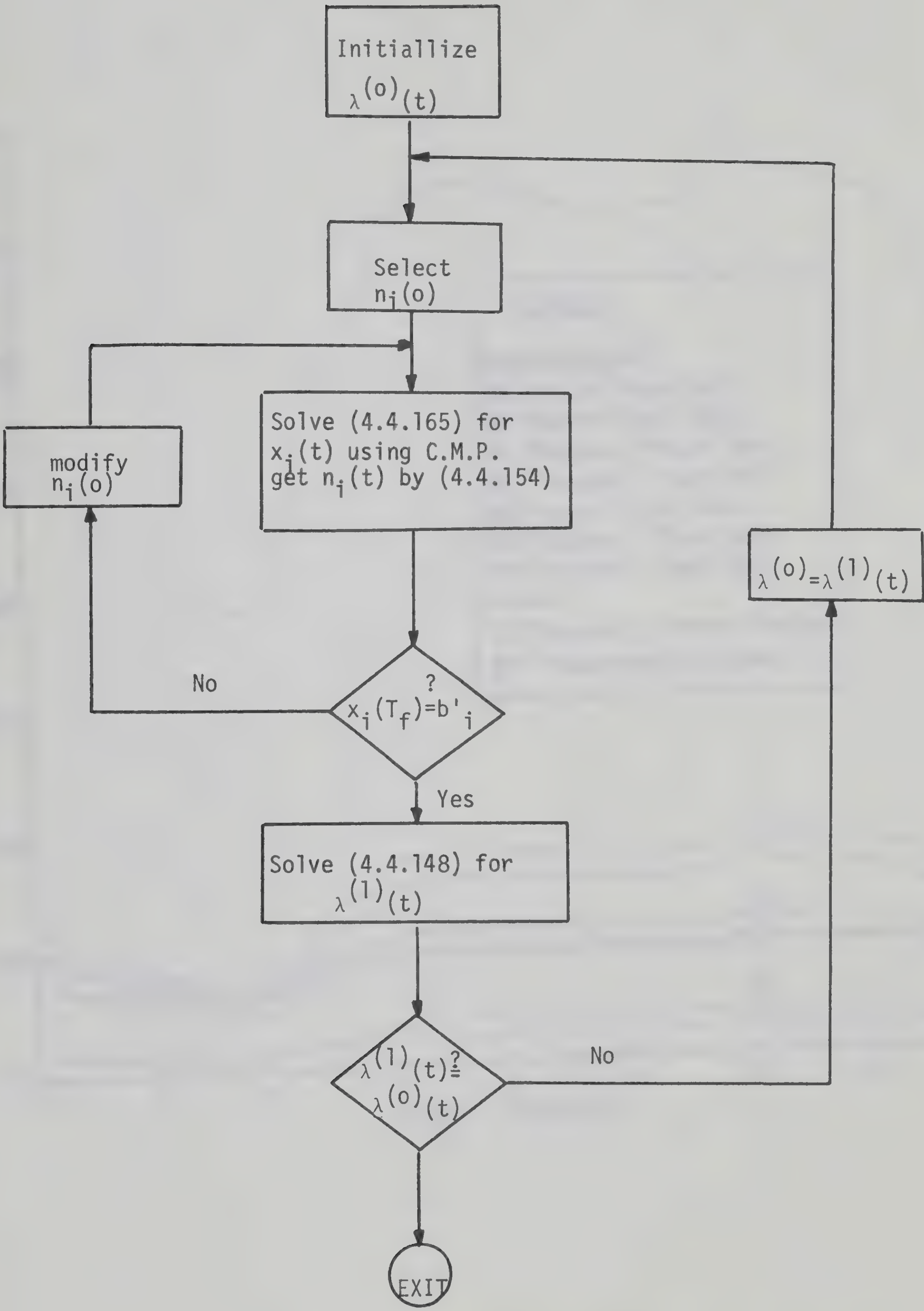


Figure 4.2 Computer Program Flow Chart.

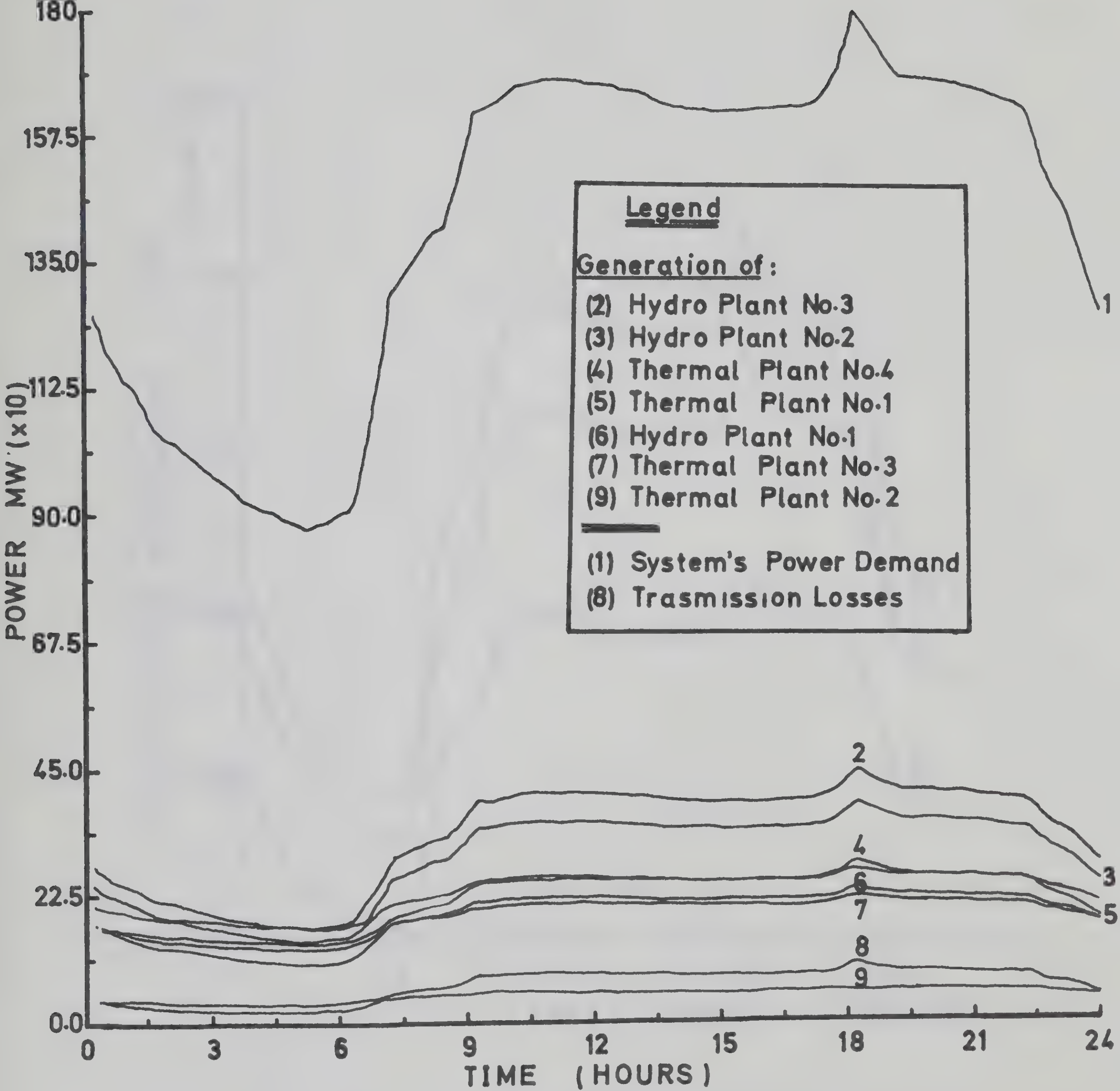


Figure 4.3a The System's Power Demand and Optimal Generations.

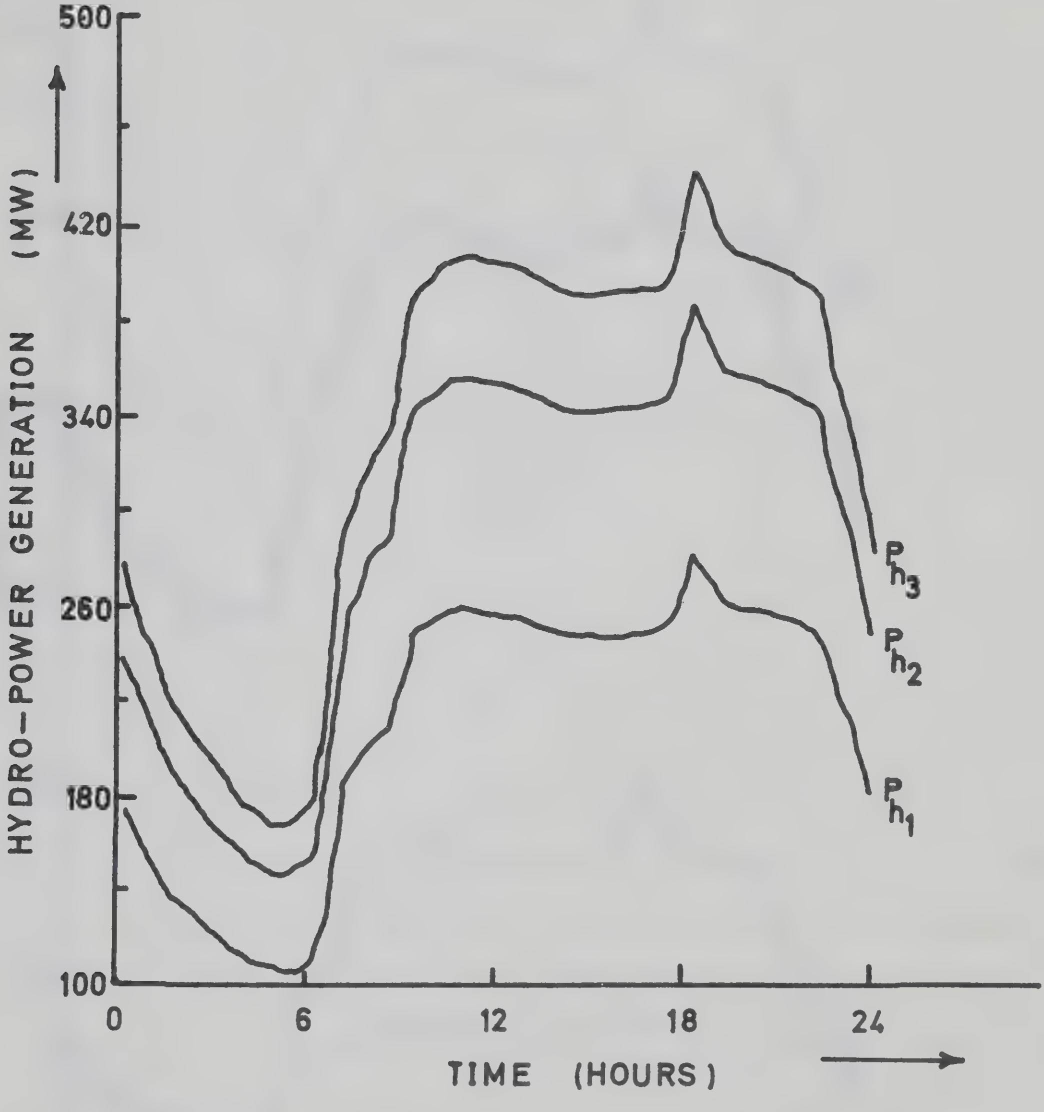


Figure 4.3b Optimal Hydro-Power Output.

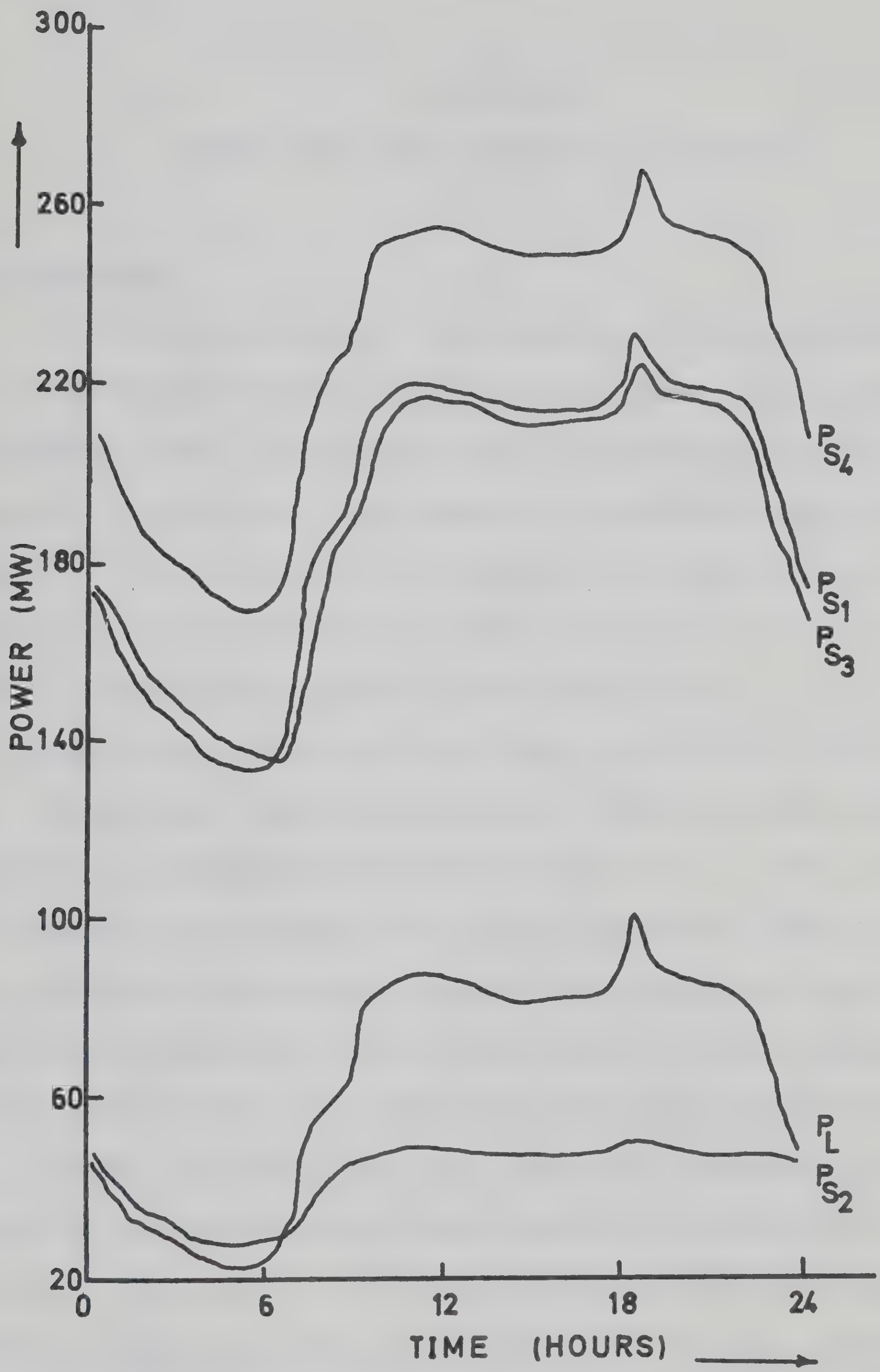


Figure 4.3c Optimal Thermal Output and Transmission Losses.

CHAPTER V

POWER SYSTEMS WITH COMMON-FLOW HYDROPLANTS

5.1 Background

The problems considered in this chapter are characterized by the presence of common-flow hydro-plants in the system. These problems are of a complex nature. The hydraulic coupling between plants on the same stream is a factor that is not present in the problems dealt with in Chapter 4. The importance of including the time delay of flow between coupled plants in the optimization scheme was pointed out by C.W. Watchorn and R.A. Arismunander in separate discussions of [7].

Among the early contributions to this problem is H.A. Burr's work [50]. He developed loading schedules for a two-plant common-flow hydro system but the assumptions made were too simplifying. Later, P.R. Menon [51] used the Euler equations for constructing sets of minimizing sequences for a three-plant hydro-thermal system. It is noted here that the system considered by Menon was of low-dimension, that is, a small number of hydro-plants were considered. Also this was a long-range scheduling problem.

In [44], E.B. Dahlin and D.W.C. Shen treat the problem of a power system with hydro plants on the same stream using the Pontryagin's Maximum Principle. They used a river flow model which introduced a large number of differential equations and boundary conditions. This was a definite contribution to the theory of economy scheduling. Unfortunately this model

made the problem more difficult to analyze numerically.

A more recent related work is that by R.H. Miller and R.P. Thompson [52]. Their work is concerned with the Pacific Gas and Electric Company hydro-thermal system. A linear programming approach is used for solving the long range scheduling problem. A set of inequality constraints on the reservoir's storage and head variations are imposed. However, the time delays of flows were not taken into consideration.

It is evident that the need still exists for an optimum scheduling scheme for common flow systems. In the following two sections two distinct problems are discussed. In Section 5.2, the problem of a system with a general number of hydro plants on the same stream is treated. A more general situation is discussed in Section 5.3. Here a system characterized by multiple chains of hydro-plants is considered. In these problems, the time delay of flow between plants on the same stream is taken into consideration. Also, the effect of the tail-race elevation on the effective hydraulic head is considered.

The time-delayed control systems field is still an active one. One of the main references on this subject is [53]. In this reference a dynamic programming approach is used to obtain the optimal solution for some certain problems. For an extension of the classical calculus of variations results to control problems with delayed arguments, reference [12] is an important contribution.

5.2 Power System with Variable Head Hydro-plants on the same stream.

An extension of the results obtained in Chapter 4 is made here. The problem of a hydro-thermal electric power system with variable head hydro-plants on the same stream is discussed in this section. The time delay of

flow between the hydro stations is taken into consideration. Also the effect of tail-race elevation on the effective hydraulic head is considered. The results of this section were reported in [54,55].

5.2.1 Statement of the Problem

An electric power system with m-thermal plants and (n-m) hydro-plants on the same stream is considered. The hydraulic part of the system is shown in Fig. (5-1). A prediction of the system's power demand and water supply is assumed available over the optimization interval. The problem is to find the active power generation of each plant as a function of time under the following conditions:

1. The total operating costs of the thermal plants over the optimization interval is a minimum.

2. The operating costs at the $i\underline{th}$ thermal plant are approximated by:

$$F_i[P_{s_i}(t)] = \alpha_i + \beta_i P_{s_i}(t) + \gamma_i P_{s_i}^{\,2}(t) \quad \$/\text{Hr} \qquad (5.2.1)$$

3. The total active generation in the system matches the load plus the losses.

4. The transmission losses in the system may be represented by the general loss formula.

5. The time integral of water discharge for each hydro-plant is a prespecified constant amount.

6. The effective hydraulic head at the $i\underline{th}$ hydro plant is equal to the difference between the forebay elevation $y_i(t)$ and the tail-race elevation $y_{T_i}(t)$, thus

$$h_i(t) = y_i(t) - y_{T_i}(t) \qquad i = m+1,\ldots,n \qquad (5.2.2)$$

7. The forebay elevation $y_i(t)$ is given by:

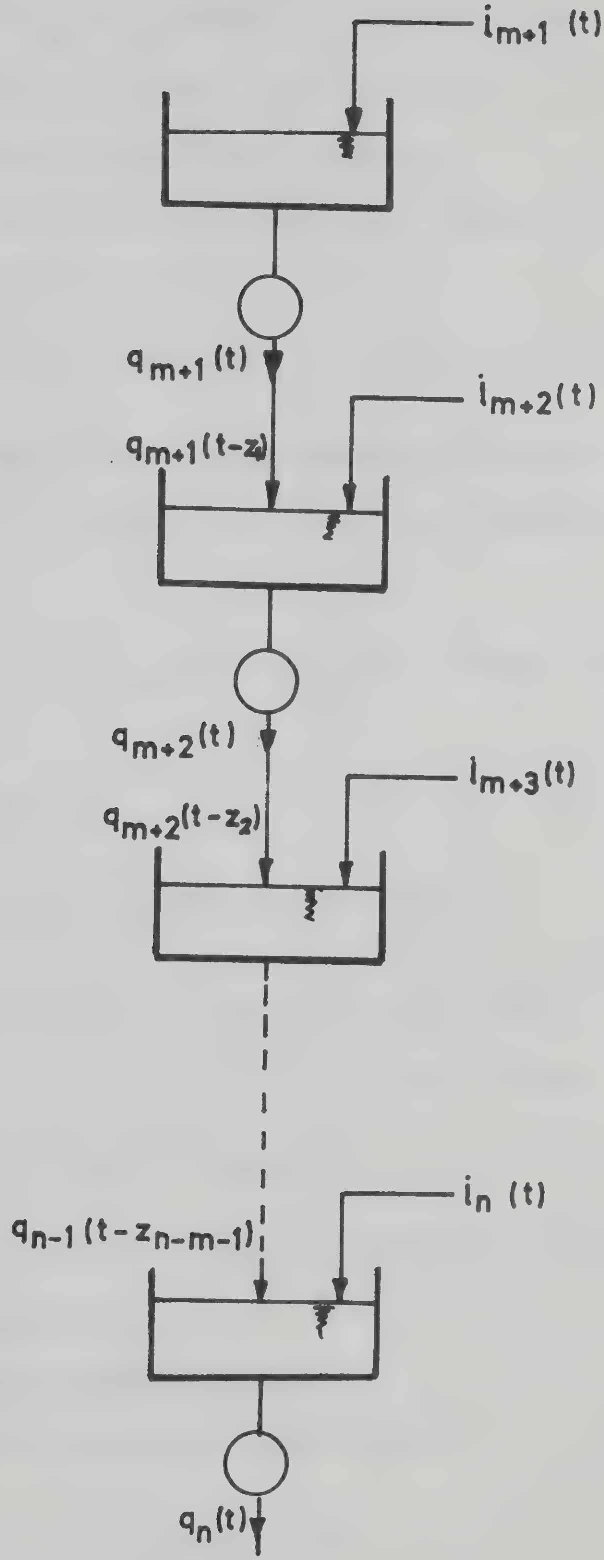


Figure 5.1 Layout of Hydro-Plants on the Same Stream.

$$y_i(t) = y_{io} + \beta_{y_i} s_i(t) \qquad i = m+1,\ldots,n \qquad (5.2.3)$$

This relation is true for vertical sided reservoirs. y_{io} and β_{y_i} are constants corresponding to the forebay geometry.

8. The tail water elevation varies with the rate of water discharge according to the relation.

$$y_{T_i}(t) = y_{T_{io}} + \beta_{T_i} q_i(t) \qquad i = m+1,\ldots,n \qquad (5.2.4)$$

$y_{T_{io}}$ and β_{T_i} are known constants corresponding to the tail-race geometry. Thus substituting (5.2.3) and (5.2.4) in (5.2.2), the effective head $h_i(t)$ is given by

$$h_i(t) = \alpha_{y_i} + \beta_{y_i} s_i(t) - \beta_{T_i} q_i(t) \quad i = m+1,\ldots,n \quad (5.2.5)$$

where

$$\alpha_{y_i} = y_{io} - y_{T_{io}} \qquad i = m+1,\ldots,n \qquad (5.2.6)$$

9. The reservoir's dynamics are described by

$$\dot{s}_{m+i}(t) = i_{m+i}(t) + q_{m+i-1}(t-\tau_{i-1}) - q_{m+i}(t)$$

$$i = 2,\ldots,(n-m) \qquad (5.2.7)$$

$$\dot{s}_{m+1}(t) = i_{m+1}(t) - q_{m+1}(t) \qquad (5.2.8)$$

Here, the time delay of water discharge between two consecutive hydro-plants is assumed to be a constant τ.

5.2.2. A Minimum Norm Formulation

The object of the optimizing computation is

$$\underset{\substack{P_{s_i}(t) \\ i=1,\ldots,m}}{\mathrm{Min}} \int_0^{T_f} \sum_{i=1}^{m} F_i[P_{s_i}(t)]dt \qquad (5.2.9)$$

The generation schedule sought must satisfy the active power balance equation.

$$P_D(t) = \sum_{i=1}^{m} P_{s_i}(t) + \sum_{i=m+1}^{n} P_{h_i}(t) - P_L(t) \qquad (5.2.10)$$

The transmission power loss is given by:

$$P_L(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} P_i(t)B_{ij}P_j(t) + \sum_{i=1}^{n} B_{io}P_i(t) + K_{Lo} \qquad (5.2.11)$$

Furthermore, the water discharge at each hydro plant is to satisfy the following constraints on the volume of water used over the optimization interval.

$$\int_0^{T_f} q_i(t)dt = b_i \qquad i = m+1,\ldots,n \qquad (5.2.12)$$

The $i\underline{th}$ hydro-plant's active power generation $P_{h_i}(t)$ is given by:

$$P_{h_i}(t) = \frac{q_i(t)h_i(t)}{G_i} \qquad i = m+1,\ldots,n \qquad (5.2.13)$$

where the G_i's are the efficiency constants of the hydro plants. Substituting (5.2.5) in (5.2.13) the hydro powers are given by:

$$P_{h_{m+i}}(t) = \frac{q_{m+i}(t)}{G_{m+i}}[\alpha y_{m+i} + \beta_{y_{m+i}} s_{m+i}(t) - \beta_{T_{m+i}} q_{m+i}(t)]$$

$$i = 1,\ldots,(n-m) \qquad (5.2.14)$$

Define the following quantities:

$$D_{m+i}(t) = s_{m+i}(o) + \int_0^t i_{m+i}(\sigma)d\sigma \qquad (5.2.15)$$

$$Y_{m+i}(t,\tau_i) = \int_0^t q_{m+i}(\sigma-\tau_i)d\sigma \qquad (5.2.16)$$

$$Q_{m+i}(t) = \int_0^t q_{m+i}(\sigma)d\sigma \qquad (5.2.17)$$

Then integrating the reservoir's dynamic equations (5.2.7) and (5.2.8) yield:

$$s_{m+1}(t) = D_{m+1}(t) - Q_{m+1}(t) \tag{5.2.18}$$

$$s_{m+i}(t) = D_{m+i}(t) + Y_{m+i-1}(t,\tau_{i-1}) - Q_{m+i}(t) \qquad i = 2,\ldots,(n-m) \tag{5.2.19}$$

Define the pseudo-control variables [54]

$$x_{m+i}(t) = s_{m+i}(t) - D_{m+i}(t) \qquad i = 2,\ldots,(n-m) \tag{5.2.20}$$

then (5.2.19) yields

$$x_{m+i}(t) = Y_{m+i-1}(t,\tau_{i-1}) - Q_{m+i}(t) \qquad i = 2,\ldots,(n-m) \tag{5.2.21}$$

Thus (5.2.14) using (5.2.20) is

$$P_{h_{m+i}}(t) = \frac{q_{m+i}(t)}{G_{m+i}} [\alpha_{m+i} + \beta_{y_{m+i}}(D_{m+i}(t) + x_{m+i}(t)) - \beta_{T_{m+i}} q_{m+i}(t)] \qquad i = 2,\ldots,(n-m) \tag{5.2.22}$$

and for the (m+1)st plant:

$$P_{h_{m+1}}(t) = \frac{q_{m+1}(t)}{G_{m+1}} [\alpha_{m+1} + \beta_{y_{m+1}}[D_{m+1}(t) - Q_{m+1}(t)] - \beta_{T_{m+1}} q_{m+1}(t)] \tag{5.2.23}$$

Let

$$A_{m+i}(t) = -\frac{1}{G_{m+i}} [\alpha_{m+i} + \beta_{y_{m+i}} D_{m+i}(t)] \qquad i = 1,\ldots,(n-m) \tag{5.2.24}$$

$$B_{m+i} = \frac{\beta_{y_{m+i}}}{G_{m+i}} \qquad i = 1,\ldots,(n-m) \quad (5.2.25)$$

$$C_{m+i} = \frac{\beta_{T_{m+i}}}{G_{m+i}} \qquad i = 1,\ldots,(n-m) \quad (5.2.26)$$

then the hydro power given by (5.2.22) and (5.2.23) becomes:

$$P_{h_{m+1}}(t) + A_{m+1}(t)q_{m+1}(t) + B_{m+1}q_{m+1}(t)Q_{m+1}(t)$$
$$+ C_{m+1}q_{m+1}^{2}(t) = 0 \qquad (5.2.27)$$

$$P_{h_{m+i}}(t) + A_{m+i}(t)q_{m+i}(t) - B_{m+i}q_{m+i}(t)x_{m+i}(t)$$
$$+ C_{m+i}q_{m+i}^{2}(t) = 0 \qquad i = 2,\ldots,(n-m) \quad (5.2.28)$$

The problem now is to minimize the cost functional given by (5.2.9) subject to satisfying (5.2.10), (5.2.16), (5.2.17), (5.2.28), (5.2.21) and (5.2.12). The constraints (5.2.21) are rewritten in the equivalent form

$$x_{m+i}^{2}(t) + Q_{m+i}^{2}(t) + 2x_{m+i}(t)Q_{m+i}(t) - Y_{m+i}^{2}(t,\tau_{i-1}) = 0$$
$$i = 2,\ldots,(n-m) \quad (5.2.29)$$

and (5.2.16) is rewritten as

$$Y_{m+i-1}(t,\tau_{i-1}) = \int_{-\tau_{i-1}}^{t-\tau_{i-1}} q_{m+i-1}(\sigma)d\sigma \qquad i = 2,\ldots,(n-m) \quad (5.2.30)$$

Let

$$\psi_{m+i-1}(t,\tau_{i-1}) = \int_{-\tau_{i-1}}^{t-\tau_{i-1}} q_{m+i-1}(\sigma)d\sigma$$
$$(t \leq \tau_{i-1}) \quad i = 2,\ldots,(n-m) \quad (5.2.31)$$

Note that $\psi_{m+i-1}(t,\tau_{i-1})$ is a known initial condition function. Then (5.2.30) becomes:

$$Y_{m+i-1}(t,\tau_{i-1}) = \psi_{m+i-1}(t,\tau_{i-1}) \qquad 0 \leq t \leq \tau_{i-1}$$

$$= \psi_{m+i-1}(\tau_{i-1},\tau_{i-1}) + Q_{m+i-1}(t-\tau_{i-1})$$

$$t_{i-1} < t \leq T_f \quad i = 2,\ldots,(n-m) \qquad (5.2.32)$$

Thus (5.2.29) reduces to

$$x_{m+i}{}^2(t) + Q_{m+i}{}^2(t) + 2x_{m+i}(t)Q_{m+i}(t) - \psi_{m+i-1}{}^2(t,\tau_{i-1}) = 0$$

$$0 \leq t \leq \tau_{i-1}$$

$$x_{m+i}{}^2(t) + Q_{m+i}{}^2(t) + 2x_{m+i}(t)Q_{m+i}(t) - [\psi_{m+i-1}{}^2(\tau_{i-1},\tau_{i-1})$$

$$+ 2\psi_{m+i-1}(\tau_{i-1},\tau_{i-1})Q_{m+i-1}(t-\tau_{i-1}) + Q_{m+i-1}{}^2(t-\tau_{i-1})] = 0$$

$$\tau_{i-1} < t \leq T_f \quad i = 2,\ldots,(n-m)$$

$$(5.2.33)$$

Moreover the constraint (5.2.17) is rewritten in the equivalent forms.

$$q_{m+1}(t) = \dot{Q}_{m+1}(t) \qquad (5.2.34)$$

and

$$q_{m+i}(t)Q_{m+i}(t) = Q_{m+i}(t)\dot{Q}_{m+i}(t)$$

$$i = 2,\ldots,(n-m) \qquad (5.2.35)$$

An augmented cost functional is formed as:

$$J(\underline{U}(t)) = \sum_{i=0}^{6} J_i(\underline{U}(t)) \qquad (5.2.36)$$

where J_o is given by (5.2.9), and $\underline{U}(t)$ is the control vector defined by:

$$\underline{U}(t) = \text{col.}[\underline{P}(t),\underline{W}_{m+1}(t),\underline{W}_{m+2}(t),\ldots,\underline{W}_n(t)] \qquad (5.2.37)$$

with

$$\underline{P}(t) = \text{col.}[P_{s_1}(t),\ldots,P_{s_m}(t),P_{h_{m+1}}(t),\ldots,P_{h_n}(t)] \tag{5.2.38}$$

$$\underline{W}_{m+1}(t) = \text{col.}[q_{m+1}(t),Q_{m+1}(t)] \tag{5.2.39}$$

$$W_{m+i}(t) = \text{col.}[q_{m+i}(t),Q_{m+i}(t),x_{m+i}(t)]$$

$$i = 2,\ldots,(n-m) \tag{5.2.40}$$

while

$$J_1[\underline{U}(t)] = \int_0^{T_f} \lambda(t)[\sum_{i=1}^{n} \sum_{j=1}^{n} P_i(t)B_{ij}P_j(t) + \sum_{i=1}^{n}(B_{io} - 1)P_i(t)]dt \tag{5.2.41}$$

$$J_2[\underline{U}(t)] = \int_0^{T_f} n_{m+1}(t)[P_{h_{m+1}}(t) + A_{m+1}(t)q_{m+1}(t) + B_{m+1}q_{m+1}(t)$$

$$Q_{m+1}(t) + C_{m+1}q_{m+1}{}^2(t)]dt \tag{5.2.42}$$

$$J_3[\underline{U}(t)] = \int_0^{T_f} [m_{m+1}(t)q_{m+1}(t) - m_{m+1}(t)\dot{Q}_{m+1}(t)]dt \tag{5.2.43}$$

$$J_4[\underline{U}(t)] = \int_0^{T_f} \sum_{i=2}^{n-m} n_{m+i}(t)[P_{h_{m+i}}(t) + A_{m+i}(t)q_{m+i}(t)$$

$$- B_{m+i}x_{m+i}(t)q_{m+i}(t) + C_{m+i}q_{m+i}{}^2(t)]dt \tag{5.2.44}$$

$$J_5[\underline{U}(t)] = \int_0^{T_f} \sum_{i=2}^{n-m} [m_{m+i}(t)q_{m+i}(t)Q_{m+i}(t) - m_{m+i}(t)\dot{Q}_{m+i}(t)$$

$$Q_{m+i}(t)]dt \tag{5.2.45}$$

$$J_6[\underline{U}(t)] = \int_0^{T_f} \sum_{i=2}^{n-m} r_{m+i}(t)[x_{m+i}{}^2(t) + Q_{m+i}{}^2(t)$$

$$+ 2x_{m+i}(t)Q_{m+i}(t) - Y_{m+i-1}{}^2(t,\tau_{i-1})]dt \tag{5.2.46}$$

where $\lambda(t)$,$n_{m+1}(t)$, $m_{m+i}(t)$ and $r_{m+i}(t)$ are unknown functions of time. J_1 corresponds to the power balance equation (5.2.10) and (5.2.11) after dropping $P_D(t)$ which is independent of the control vector $\underline{U}(t)$. J_2 and J_4 correspond to the constraints (5.2.27) and (5.2.28). J_3 and J_5 correspond to (5.2.34) and (5.2.35) respectively. Finally J_6 corresponds to (5.2.29). Note that in (5.2.45) and (5.2.46) equivalent forms of the constraints (5.2.17) and (5.2.21) were used so that a valid norm definition can be obtained.

It is more convenient to reduce J_2, J_3 and J_5 to modified values by performing integration by parts and neglecting terms explicitly independent of the control $\underline{U}(t)$. Then the following is obtained

$$J_2[\underline{U}(t)] = \int_0^{T_f} \{n_{m+1}(t)[P_{h_{m+1}}(t) + A_{m+1}(t)q_{m+1}(t) + C_{m+1} q_{m+1}{}^2(t)] - \frac{B_{m+1}\dot{n}_{m+1}(t)}{2} Q_{m+1}{}^2(t)\}dt \tag{5.2.47}$$

$$J_3[\underline{U}(t)] = \int_0^{T_f} [m_{m+1}(t)q_{m+1}(t) + \dot{m}_{m+1}(t)Q_{m+1}(t)]dt \tag{5.2.48}$$

$$J_5[\underline{U}(t)] = \int_0^{T_f} \sum_{i=2}^{n-m} [m_{m+i}(t)q_{m+i}(t)Q_{m+i}(t) + \frac{\dot{m}_{m+i}}{2} Q_{m+i}{}^2(t)]dt \tag{5.2.49}$$

Let

$$J_6[\underline{U}(t)] = J_{6_{X,Q}}[\underline{U}(t)] - J_{6_Y}[\underline{U}(t)] \tag{5.2.50}$$

with

$$J_{6_{X,Q}}[\underline{U}(t)] = \int_0^{T_f} \sum_{i=2}^{n-m} r_{m+i}(t)[x_{m+i}{}^2(t)Q_{m+i}{}^2(t) + 2x_{m+i}(t)Q_{m+i}(t)]dt \tag{5.2.51}$$

$$J_{6_\gamma}[\underline{U}(t)]=\int_0^{T_f} \sum_{i=2}^{n-m} r_{m+i}(t)\gamma_{m+i-1}^2(t,\tau_{i-1})dt \tag{5.2.52}$$

Interchanging the integration and summation operations yields

$$J_{6_\gamma}[\underline{U}(t)] = \sum_{i=2}^{n-m} \int_0^{T_f} r_{m+i}(t)\gamma_{m+i-1}^2(t,\tau_{i-1})dt \tag{5.2.53}$$

The last equation can also be written as:

$$\begin{aligned} J_{6_\gamma}[\underline{U}(t)]= \sum_{i=2}^{n-m} [&\int_0^{\tau_{i-1}} r_{m+i}(t)\psi_{m+i-1}^2(t,\tau_{i-1})dt \\ &+\int_{\tau_{i-1}}^{T_f} r_{m+i}(t)[\psi_{m+i-1}^2(\tau_{i-1},\tau_{i-1}) \\ &+ 2\psi_{m+i-1}(\tau_{i-1},\tau_{i-1})Q_{m+i-1}(t-\tau_{i-1}) \\ &+ Q_{m+i-1}^2(t-\tau_{i-1})]dt] \end{aligned} \tag{5.2.54}$$

Dropping terms explicitly independent of the control, it is only necessary to consider

$$\begin{aligned} J'_{6_\gamma}[\underline{U}(t)] = \sum_{i=2}^{n-m} \int_{\tau_{i-1}}^{T_f} & r_{m+i}(t)[2\psi_{m+i-1}(\tau_{i-1},\tau_{i-1}) \\ & Q_{m+i-1}(t-\tau_{i-1}) + Q_{m+i-1}^2(t-\tau_{i-1})]dt \end{aligned}$$

Let $s = t-\tau_{i-1}$

then

$$J'_{6_\gamma}[\underline{U}(t)] = \sum_{i=2}^{n-m} \int_0^{T_f-\tau_{i-1}} r_{m+i}(s+\tau_{i-1})[2\psi_{m+i-1}(\tau_{i-1},\tau_{i-1})\cdot$$

$$Q_{m+i-1}(s) + Q_{m+i-1}{}^2(s)]ds \qquad (5.2.55)$$

Define

$$\begin{aligned} \phi_{m+i}(t,\tau_{i-1}) &= r_{m+i}(t+\tau_{i-1}) && t\ \varepsilon[0,T_f-\tau_{i-1}] \\ &= 0 && t\ \varepsilon[T_f-\tau_{i-1},T_f] \qquad (5.2.56) \end{aligned}$$

$$\begin{aligned} \hat{\phi}_{m+i}(t,\tau_{i-1}) &= 2\psi_{m+i-1}(\tau_{i-1},\tau_{i-1})r_{m+i}(t+\tau_{i-1}) \\ & \qquad\qquad t\ \varepsilon[0,T_f-\tau_{i-1}] \\ &= 0 \qquad\qquad t\ \varepsilon[T_f-\tau_{i-1},T_f] \qquad (5.2.57) \end{aligned}$$

Using these definitions, (5.2.55) becomes:

$$J'_{6_\gamma}[\underline{U}(t)] = \sum_{i=2}^{n-m} \int_0^{T_f} [\hat{\phi}_{m+i}(t,\tau_{i-1})Q_{m+i-1}(t)$$

$$+ \phi_{m+i}(t,\tau_{i-1})Q_{m+i-1}{}^2(t)]dt$$

or

$$J'_{6_\gamma}[\underline{U}(t)] = \int_0^{T_f} \sum_{i=1}^{n-m-1} [\hat{\phi}_{m+i+1}(t,\tau_i)Q_{m+i}(t)$$

$$+ \phi_{m+i+1}(t,\tau_i)Q_{m+i}{}^2(t)]dt \qquad (5.2.58)$$

Thus J_6 given by (5.2.50) reduces to

$$J_6[\underline{U}(t)] = \int_0^{T_f} [\sum_{i=2}^{n-m-1} (r_{m+i}(t) - \phi_{m+i+1}(t,\tau_i))Q_{m+i}{}^2(t)$$

$$-\phi_{m+2}(t,\tau_1)Q_{m+1}{}^2(t) + r_n(t)Q_n^2(t)$$

$$+ \sum_{i=2}^{n-m} r_{m+i}(t)\{x_{m+i}{}^2(t) + 2x_{m+1}(t)$$

$$Q_{m+i}(t)\} - \sum_{i=1}^{n-m-1} \dot{p}_{m+i+1}(t,\tau_i)Q_{m+i}(t)]dt \tag{5.2.59}$$

Integrating the last term of the integrand in (5.2.59) and neglecting terms explicitly independent of the control, one obtains

$$J_6[\underline{U}(t)] = \int_0^{T_f} [\sum_{i=2}^{n-m-1} (r_{m+i}(t) - \phi_{m+i+1}(t,\tau_i))Q_{m+i}{}^2(t)$$

$$- \phi_{m+2}(t,\tau_1)Q_{m+1}{}^2(t) + r_n(t)Q_n^2(t)$$

$$+ \sum_{i=2}^{n-m} r_{m+i}(t)\{x_{m+i}{}^2(t) + 2x_{m+i}(t)\ Q_{m+i}(t)\}$$

$$+ \sum_{i=2}^{n-m-1} p_{m+i+1}(t,\tau_i)q_{m+i}(t) - \dot{p}_{m+2}(t,\tau_1)$$

$$Q_{m+1}(t)]dt \tag{5.2.60}$$

Substituting (5.2.9), (5.2.41), (5.2.47), (5.2.48), (5.2.44), (5.2.49) and (5.2.60) in (5.2.36) for J_i, $i = 0,\ldots,6$ respectively yields

$$J[\underline{U}(t)] = J_L[\underline{U}(t)] + J_Q[\underline{U}(t)]$$

with

$$J_L[\underline{U}(t)] = \int_0^{T_f} [\sum_{i=1}^{m} [\beta_i - \lambda(t)\{1 - B_{io}\}]P_{s_i}(t)$$

$$+ \sum_{i=m+1}^{n} [n_i(t) - \lambda(t)\{1 - B_{io}\}]P_{h_i}(t)$$

$$+ [n_{m+1}(t)A_{m+1}(t) + m_{m+1}(t)]q_{m+1}(t)$$

$$+ [\dot{m}_{m+1}(t) - \dot{p}_{m+2}(t,\tau_1)]Q_{m+1}(t)$$

$$+ \sum_{i=2}^{n-m-1} [n_{m+i}(t)A_{m+i}(t) + p_{m+i+1}(t,\tau_i)q_{m+i}(t)$$

$$+ n_n(t)A_n(t)q_n(t) + \sum_{i=2}^{n-m}(o)Q_{m+i}(t)$$

$$+ \sum_{i=2}^{n-m}(o)x_{m+i}(t)]dt \qquad (5.2.61)$$

and

$$J_Q[\underline{U}(t)] = \int_0^{T_f} \{\sum_{i=1}^{m} \gamma_i P_{s_i}{}^2(t) + \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda(t)P_i(t)$$

$$B_{ij}P_j(t) + \sum_{i=1}^{n-m} C_{m+i}n_{m+i}(t)q_{m+i}{}^2(t)$$

$$- [\phi_{m+2}(t,\tau_1) + \frac{B_{m+1}\dot{n}_{m+1}(t)}{2}]Q_{m+1}{}^2(t)$$

$$+ (\sum_{i=2}^{n-m-1} \{\frac{\dot{m}_{m+i}(t)}{2} + r_{m+i}(t) - \phi_{m+i+1}(t,\tau_i)\}$$

$$Q_{m+i}{}^2(t)) + (\frac{\dot{m}_n(t)}{2} + r_n(t))Q_n^2(t)$$

$$+ \sum_{i=2}^{n-m}[r_{m+i}(t)\{x_{m+i}{}^2(t) + 2x_{m+i}(t)$$

$$Q_{m+i}(t) + m_{m+i}(t)q_{m+i}(t)Q_{m+i}(t)$$

$$- B_{m+i}n_{m+i}(t)q_{m+i}(t)x_{m+i}(t)]\}dt \qquad (5.2.62)$$

Define

$$\underline{L}(t) = \text{col.}[\underline{L}_p(t),\underline{L}_{m+1}(t),\underline{L}_{m+2}(t),\ldots,\underline{L}_n(t)] \qquad (5.2.63)$$

where

$$\underline{L}_p(t) = \text{col.}[\ell_{p_{s_1}}(t),\ldots,\ell_{p_{s_m}}(t),\ell_{p_{h_{m+1}}}(t),\ldots,\ell_{p_{h_n}}(t)]$$

$$(5.2.64)$$

$$\underline{L}_{m+1}(t) = \text{col.}[\ell_{(m+1)q}(t), \ell_{(m+1)Q}(t)] \quad (5.2.65)$$

$$\underline{L}_{m+i}(t) = \text{col.}[\ell_{(m+i)q}(t), \ell_{(m+i)Q}(t), \ell_{(m+i)x}(t)] \quad (5.2.66)$$

$$i = 2,\ldots,n-m$$

and the square symmetric matrix $\underline{B}(t)$ as:

$$\underline{B}(t) = \text{diag}[\underline{B}_p(t), \underline{B}_{m+1}(t), \underline{B}_{m+2}(t), \ldots, \underline{B}_n(t)] \quad (5.2.67)$$

with

$$\ell_{p_{s_i}}(t) = \beta_i - \lambda(t)[1 - B_{io}] \quad (5.2.68)$$

$$\ell_{p_{hm+i}}(t) = n_{m+i}(t) - \lambda(t)[1 - B_{(m+i)o}] \quad (5.2.69)$$

$$\ell_{(m+1)q}(t) = A_{m+1}(t)n_{m+1}(t) + m_{m+1}(t) \quad (5.2.70)$$

$$\ell_{(m+1)Q}(t) = \dot{m}_{m+1}(t) - \dot{p}_{m+2}(t,\tau_1) \quad (5.2.71)$$

$$\ell_{(m+i)q}(t) = A_{m+i}(t)n_{m+i}(t) + p_{m+i+1}(t,\tau_i) \quad (5.2.72)$$

$$i = 2,\ldots,n-m-1$$

$$\ell_{(m+i)Q}(t) = 0 \qquad i = 2,\ldots,n-m \quad (5.2.73)$$

$$\ell_{(m+i)x}(t) = 0 \qquad i = 2,\ldots,n-m \quad (5.2.74)$$

$$\ell_{nq}(t) = A_n(t)n_n(t) \quad (5.2.75)$$

$$\underline{B}_p(t) = \begin{bmatrix} \gamma_1 + B_{11}\lambda(t) & B_{12}\lambda(t) & \cdots\cdots & B_{1n}\lambda(t) \\ B_{12}\lambda(t) & \gamma_2 + B_{22}\lambda(t) & \cdots\cdots & B_{2n}\lambda(t) \\ \vdots & \vdots & & \\ \vdots & \cdots\cdots B_{(m+1)(m+1)}\lambda(t) & & \\ \vdots & & & B_{nn}\lambda(t) \end{bmatrix} \quad (5.2.76)$$

$$\underline{B}_{m+1}(t) = \text{diag}[C_{m+1}n_{m+1}(t), -(\phi_{m+2}(t,\tau_1) + B_{m+1}\frac{\dot{n}_{m+1}(t)}{2})]$$

$$(5.2.77)$$

$$\underline{B}_{m+i}(t) = (b^{j,k}_{(m+i)}(t)) \qquad i = 2,\ldots,n\text{-}m \qquad (5.2.78)$$

$$b^{11}_{m+i}(t) = C_{m+i}n_{m+i}(t) \qquad i = 2,\ldots,n\text{-}m \qquad (5.2.79)$$

$$b^{12}_{m+i}(t) = \frac{m_{m+i}(t)}{2} \qquad i = 2,\ldots,n\text{-}m \qquad (5.2.80)$$

$$b^{13}_{m+i}(t) = -\frac{B_{m+i}n_{m+i}(t)}{2} \qquad i = 2,\ldots,n\text{-}m \qquad (5.2.81)$$

$$b^{22}_{m+i}(t) = \frac{\dot{m}_{m+i}(t)}{2} + r_{m+i}(t) - \phi_{m+i+1}(t,\tau_i)$$

$$i = 2,\ldots,n\text{-}m\text{-}1 \qquad (5.2.82)$$

$$b^{23}_{m+i}(t) = r_{m+i}(t) \qquad i = 2,\ldots,n\text{-}m \qquad (5.2.83)$$

$$b^{33}_{m+i}(t) = r_{m+i}(t) \qquad i = 2,\ldots,n\text{-}m \qquad (5.2.84)$$

$$b^{22}_{n}(t) = \frac{\dot{m}_n(t)}{2} + r_n(t) \qquad (5.2.85)$$

This reduces $J[\underline{U}(t)]$ to:

$$J[\underline{U}(t)] = \int_0^{T_f} [\underline{L}^T(t)\underline{u}(t) + \underline{u}^T(t)\underline{B}(t)\underline{u}(t)]dt \qquad (5.2.86)$$

Let $\qquad \underline{V}^T(t) = \underline{L}^T(t)\underline{B}^{-1}(t) \qquad (5.2.87)$

then (5.2.86) is rewritten as

$$J[\underline{U}(t)] = \int_0^{T_f} \{[\underline{U}(t) + \frac{\underline{V(t)}}{2}]^T\underline{B}(t)[\underline{U}(t) + \frac{\underline{V(t)}}{2}]$$

$$- \frac{\underline{V}^T(t)}{2}\,\underline{B}(t)\,\frac{\underline{V(t)}}{2}\}dt$$

The last term in the integrand of the last expression does not depend explicitly on the control $\underline{U}(t)$. Thus one needs only to consider minimizing:

$$J[\underline{U}(t)] = \int_0^{T_f} [\underline{U}(t) + \frac{\underline{V}(t)}{2}]^T \underline{B}(t)[\underline{U}(t) + \frac{\underline{V}(t)}{2}]dt \tag{5.2.88}$$

subject to

$$\int_0^{T_f} q_{m+i}(\sigma)d\sigma = b_{m+i} \qquad i = 1,\ldots,n-m \tag{5.2.89}$$

Define the (n-m)x1 column vector:

$$\underline{b} = \text{col.}[b_{m+1},\ldots,b_n] \tag{5.2.90}$$

and (n-m)x(4n-3m-1) matrix K^T as:

$$\underline{K}^T = \begin{bmatrix} \underline{0} & \underline{K}^T_{m+1} & \underline{0} & & \\ \underline{0} & \underline{0} & \underline{K}^T_{m+2} & & \\ & & & & \\ & & & & \underline{K}^T_n \\ & & & & \end{bmatrix}$$

(5.2.91a)

with $\quad \underline{K}^T_{m+1} = [1\ 0]$ (5.2.91b)

$K^T_{m+i} = [1\ 0\ 0]$ (5.2.91c)

This transforms the constraint (5.2.89) to:

$$\underline{b} = \int_0^{T_f} \underline{K}^T \underline{u}(s)ds \tag{5.2.92}$$

The control vector $\underline{U}(t)$ is considered an element of the Hilbert space $L_{2,\underline{B}}^{[4n-3m-1]}[0,T_f]$ of the [4n-3m-1] vector valued square integrable functions defined on $[0,T_f]$ endowed with the inner product definition:

$$<\underline{V}(t),\underline{u}(t)> = \int_0^{T_f} \underline{V}^T(t)\underline{B}(t)\underline{u}(t)dt \tag{5.2.93}$$

for every $\underline{V}(t)$ and $\underline{u}(t)$ in $L_{2,B}^{[4n-3m-1]}[0,T_f]$, provided that $\underline{B}(t)$ is positive definite.

The given vector $\underline{b}$ is considered an element of the Real Space $R^{(n-m)}$ with the Euclidean inner product definition:

$$<\underline{X},\underline{Y}> = \underline{X}^T\underline{Y} \tag{5.2.94}$$

for every $\underline{X}$ and $\underline{Y}$ in $R^{(n-m)}$.

Equation (5.2.92) defines a bounded linear transformation $T: L_{2,\underline{B}}^{[4n-3m-1]}[0,T_f]\to R^{n-m}$. This can be writen as:

$$\underline{b} = T[\underline{u}(t)] \tag{5.2.95}$$

and the cost functional given in (5.2.88) reduces to:

$$J[\underline{u}(t)]= ||\underline{u}(t) + \frac{\underline{V}(t)}{2}||^2 \tag{5.2.96}$$

Finally, it is necessary only to minimize

$$J[\underline{u}(t)] = ||\underline{u}(t) + \frac{\underline{V}(t)}{2}|| \tag{5.2.97}$$

subject to $\underline{b} = T[\underline{u}(t)]$

for a given $\underline{b}$ in $R^{(n-m)}$

5.2.3 The Optimal Solution

The optimal solution to the problem formulated in the previous subsection, using the results of Chapter 2 is:

$$\underline{u}_{\xi}(t) = T^{\dagger}[\underline{b} + T(\frac{\underline{V}(t)}{2})] - \frac{\underline{V}(t)}{2} \tag{5.2.98}$$

where $T^{\dagger}$ is obtained as follows:

T^*, the adjoint of T, is obtained using the identity:

$$<\underline{\xi},\underline{Tu}>_{R^{n-m}} = <\underline{T^*\xi},\underline{u}>_{L_{2,B}^{(4n-3m-1)}[0,T_f]} \tag{5.2.99}$$

Let

$$\underline{\xi} = \text{col.}[\xi_{m+1},\ldots,\xi_n] \tag{5.2.100}$$

$$\underline{T^*\xi} = \text{col.}[\underline{T}_p,\underline{T}_{m+1},\underline{T}_{m+2},\ldots,\underline{T}_n] \tag{5.2.101}$$

$$\underline{T}_p = \text{col}[t_1,\ldots,t_n] \tag{5.2.102}$$

$$\underline{T}_{m+1} = \text{col}.[t_{q_{m+1}}, t_{Q_{m+1}}] \tag{5.2.103}$$

$$\underline{T}_{m+i} = \text{col}.[t_{q_{m+i}}, t_{Q_{m+i}}, t_{x_{m+i}}] \tag{5.2.104}$$

In $R^{(n-m)}$, the inner product of the left-handside of (5.2.99) is:

$$\langle \xi, \underline{Tu} \rangle = \underline{\xi}^T \int_0^{T_f} \underline{K}^T \underline{u}(s)ds \tag{5.2.105}$$

This reduces to

$$\langle \underline{\xi}, \underline{Tu} \rangle = \sum_{i=m+1}^{n} \xi_i \int_0^{T_f} q_i(\sigma)d\sigma \tag{5.2.106}$$

In $L_{2,B}^{[4n-3m-1]}[0,T_f]$, the inner product of the right-handside in (5.2.99) is:

$$\langle \underline{T^*\xi}, \underline{u} \rangle = \int_0^{T_f} (\underline{T^*\xi})^T \underline{B}(t)\underline{u}(t)dt \tag{5.2.107}$$

Using (5.2.37), (5.2.67) and (5.2.102) this reduces to

$$\langle \underline{T^*\xi}, \underline{u} \rangle = \int_0^{T_f} [\underline{T}_p{}^T \underline{B}_p(t)\underline{P}(t) + \underline{T}_{m+1}^T \underline{B}_{m+1}(t)\underline{W}_{m+1}(t)$$

$$+ \sum_{i=2}^{n-m} \underline{T}_{m+i}^T \underline{B}_{m+i}(t)\underline{W}_{m+i}(t)]dt \tag{5.2.108}$$

Thus the identity of (5.2.99) using (5.2.106) and (5.2.108) yields:

$$\int_0^{T_f} \sum_{i=1}^{n-m} b_{m+i} q_{m+i}(\sigma)d\sigma = \int_0^{T_f} [\underline{T}_p{}^T \underline{B}_p(t)P(t) + \sum_{i=1}^{n-m} T^T_{-m+i} \underline{B}_{m+i}(t)$$

$$\underline{W}_{m+i}(t)]dt \tag{5.2.109}$$

Let

$$\phi_{m+1}^T = [b_{m+1}\ 0] \tag{5.2.110}$$

$$\phi^T_{m+i} = [b_{m+i} \quad 0 \quad 0] \qquad i = 2,\ldots,n-m \tag{5.2.111}$$

then (5.2.109) becomes

$$\int_0^{T_f} \sum_{i=1}^{n-m} \phi^T_{m+i} \underline{W}_{m+i}(\sigma)d\sigma = \int_0^{T_f} [\underline{T}^T_p \underline{B}_p(t)P(t) + \sum_{i=1}^{n-m} \underline{T}^T_{m+i}\underline{B}_{m+i}(t)$$

$$\underline{W}_{m+i}(t)]dt \tag{5.2.112}$$

Equation (5.2.112) is satisfied for:

$$\underline{T}^T_p(t) = \underline{0} \tag{5.2.113}$$

$$T^T_{m+i}(t) = \underline{\phi}^T_{m+i} \underline{B}^{-1}_{m+i} \qquad i = 1,\ldots,n-m \tag{5.2.114}$$

From (5.2.77) the following is obtained:

$$\underline{B}^{-1}_{m+1}(t) = \text{diag}[\{C_{m+1}n_{m+1}(t)\}^{-1}, \{-(\phi_{m+2}(t,\tau_1)$$

$$+ B_{m+1} \frac{\dot{n}_{m+1}(t)}{2})\}^{-1}] \tag{5.2.115}$$

thus one obtains from (5.2.114) for i = 1 and (5.2.110):

$$T^T_{m+1}(t) = [\frac{b_{m+1}}{C_{m+1}n_{m+1}(t)}, 0] \tag{5.2.116}$$

Let

$$B^{-1}_{m+i}(t) = [a_{ij_{m+i}}(t))] \qquad i = 2,\ldots,(n-m) \tag{5.2.117}$$

then by (5.2.114) and (5.2.111) one obtains:

$$\underline{T}^T_{m+i}(t) = [b_{m+i}a_{11_{m+i}}(t), b_{m+i}a_{12_{m+i}}(t), b_{m+i}a_{13_{m+i}}(t)]$$

$$i = 2,\ldots,n-m \tag{5.2.118}$$

This means now that $\underline{T^*\xi}$ as given by (5.2.101) is completely defined

by (5.2.113), (5.2.116) and (5.2.118).

The operator J is next evaluated as:

$$J(\underline{\xi}) = T[\underline{T^*\xi}] \qquad (5.2.119)$$

Using (5.2.91), (5.2.92), (5.2.95) and (5.2.101) this is given by:

$$J(\underline{\xi}) = \text{col.}[b_{m+1}\int_0^{T_f} \frac{1}{C_{m+1}n_{m+1}(t)}\,dt, b_{m+2}\int_0^{T_f} a_{11_{m+2}}(t)dt,\ldots] \qquad (5.2.120)$$

or

$$J[\underline{\xi}] = \underline{\Lambda}\ \underline{\xi} \qquad (5.2.121)$$

with

$$\Lambda = \text{diag}[\int_0^{T_f} \frac{1}{C_{m+1}n_{m+1}(t)}\,dt, \int_0^{T_f} a_{11_{m+2}}(t)dt,\ldots] \qquad (5.2.122)$$

Thus the inverse operation J^{-1} is given by:

$$J^{-1}[\underline{\xi}] = \underline{\Lambda}^{-1}\ \underline{\xi} \qquad (5.2.123)$$

where

$$\Lambda^{-1} = \text{diag}[(\int_0^{T_f}[1/C_{m+1}n_{m+1}(t)]dt)^{-1},(\int_0^{T_f} a_{11_{m+2}}(t)dt)^{-1},\ldots] \qquad (5.2.124)$$

Finally, the pseudo-inverse operator $T^\dagger$ is obtained from the definition:

$$\underline{T^\dagger\xi} = T^*[\underline{J^{-1}\xi}]$$

Using (5.2.101), (5.2.113) and (5.2.118) this becomes

$$\underline{T^\dagger\xi} = \text{col.}[\underline{t}_p^\dagger,\underline{t}_{W_{m+1}}^\dagger,\ldots,t_{W_n}^\dagger] \qquad (5.2.125)$$

with

$$\underline{t}_p^\dagger = \text{col.}[0,\ldots,0] \qquad (5.2.126)$$

$$\underline{t}_{W_{m+1}}^\dagger = \text{col.}[\frac{b_{m+1}}{C_{m+1}n_{m+1}(t)\int_0^{T_f}(1/C_{m+1}n_{m+1}(t))dt}, 0] \qquad (5.2.127)$$

$$\underline{t}^{\dagger}_{W_{m+i}} = \text{col.}[t^{\dagger}_{(m+i)_1}, t^{\dagger}_{(m+i)_1} \frac{a_{12_{(m+i)}}(t)}{a_{11_{(m+i)}}(t)},$$

$$t^{\dagger}_{(m+i)_1} \frac{a_{13_{(m+i)}}(t)}{a_{11_{(m+i)}}(t)}] \quad i = 2,\dots,n\text{-}m$$

(5.2.128)

$$t^{\dagger}_{m+i} = \frac{b_{m+i}a_{11_{(m+i)}}(t)}{\int_{o}^{T_f} a_{11_{(m+i)}}(t)dt} \tag{5.2.129}$$

Let

$$\underline{n} = \underline{b} + T(\frac{\underline{V}}{2}) \tag{5.2.130}$$

and

$$\underline{n} = \text{col.}[n_{m+1},\dots,n_n] \tag{5.2.131}$$

then applying (5.2.92) one obtains:

$$n_{m+i} = b_{m+i} + \int_{o}^{T_f} \frac{1}{2}V_{(m+i)_1}(t)dt \tag{5.2.132}$$

where it is assumed that $\underline{V}(t)$ given by (5.2.87) is obtained in the partioned form:

$$\underline{V}^T(t) = [\underline{V}_p^T(t),\underline{V}_{m+1}^T(t),\dots,\underline{V}_n^T(t)] \tag{5.2.133}$$

$$\underline{V}_p^T(t) = [V_{p_{s_1}}(t),\dots,V_{p_{h_{m+1}}}(t),\dots,V_{p_{h_n}}(t)] \tag{5.2.134}$$

$$V_{m+1}^T(t) = [V_{(m+1)_1}(t),V_{(m+1)_2}(t)] \tag{5.2.135}$$

$$V_{m+i}^T(t) = [V_{(m+i)_1}(t),V_{(m+i)_2},V_{(m+i)_3}] \tag{5.2.136}$$

$$i = 2,\dots,(n\text{-}m)$$

This gives the optimal solution of equation (5.2.98) in component form as:

$$\underline{P}_{\xi}(t) = -\,\underline{V}_p(t)/2 \tag{5.2.137}$$

$$q_{\xi_{m+1}}(t) = -\frac{V_{(m+1)_1}(t)}{2} + \frac{b_{m+1} + \int_0^{T_f} \frac{V_{(m+1)_1}(t)}{2}\,dt}{n_{m+1}(t)\int_0^{T_f}[1/n_{m+1}(t)]dt} \tag{5.2.138}$$

$$Q_{\xi_{m+1}}(t) = -\frac{V_{(m+1)_2}(t)}{2} \tag{5.2.139}$$

$$q_{\xi_{m+i}}(t) = -\frac{V_{(m+i)_1}(t)}{2} + \frac{\eta_{m+i}a_{11_{(m+i)}}(t)}{\int_0^{T_f} a_{11_{(m+i)}}(t)dt}$$

$$i = 2,\ldots,n-m \tag{5.2.140}$$

$$Q_{\xi_{m+i}}(t) = -\frac{V_{(m+i)_2}(t)}{2} + \frac{\eta_{m+i}a_{12_{(m+i)}}(t)}{\int_0^{T_f} a_{11_{(m+i)}}(t)dt}$$

$$i = 2,\ldots,n-m \tag{5.2.141}$$

$$x_{\xi_{m+i}}(t) = -\frac{V_{(m+i)_3}(t)}{2} + \frac{\eta_{m+i}a_{13_{(m+i)}}(t)}{\int_0^{T_f} a_{11_{(m+i)}}(t)dt} \tag{5.2.142}$$

5.2.4. The Modified Optimal Solution

In formulating the problem at hand, pseudo control variables $q(t)$ and $x(t)$ were introduced. It is possible to eliminate these variables together with the multiplier functions $m(t)$ and $r_i(t)$ associated with them. First we establish some relations between some of the variables encountered in the previous subsection.

Using (5.2.66), (5.2.72), (5.2.73), (5.2.74), (5.2.87) and (5.2.117) in (5.2.136) we obtain:

$$V_{(m+i)_1}(t) = [A_{m+i}(t)n_{m+i}(t) + p_{m+i+1}(t,\tau_i)]a_{11_{(m+i)}}(t)$$

$$i = 2,\ldots,n-m \qquad (5.2.143)$$

$$V_{(m+i)_2}(t) = V_{(m+i)_1}(t)\,\frac{a_{12_{(m+i)}}(t)}{a_{11_{(m+i)}}(t)}$$

$$i = 2,\ldots,(n-m) \qquad (5.2.144)$$

$$V_{(m+i)_3}(t) = V_{(m+i)_1}(t)\,\frac{a_{13_{m+i}}(t)}{a_{11_{(m+i)}}(t)}$$

$$i = 2,\ldots,n-m \qquad (5.2.145)$$

substituting (5.2.144) and (5.2.145) in (5.2.141) and using (5.2.140) respectively one finds:

$$Q_{\xi_{m+i}}(t) = \frac{a_{12_{(m+i)}}(t)}{a_{11_{(m+i)}}(t)}\; q_{\xi_{m+i}}(t) \qquad i = 2,\ldots,n-m \quad (5.2.146)$$

$$x_{\xi_{m+i}}(t) = \frac{a_{13_{(m+i)}}(t)}{a_{11_{(m+i)}}(t)}\; q_{\xi_{m+i}}(t) \qquad i = 2,\ldots,n-m \quad (5.2.147)$$

Invoking the constraint equation (5.2.21) using (5.2.146) and (5.2.147) one obtains:

$$Y_{m+i-1}(t,\tau_{i-1}) = \frac{a_{12_{(m+i)}}(t) + a_{13_{(m+i)}}(t)}{a_{11_{(m+i)}}(t)}\; q_{\xi_{m+i}}(t)$$

$$i = 2,\ldots,n-m \quad (5.2.148)$$

From (5.2.78) through (5.2.85) and (5.2.117) it follows that

$$a_{11_{(m+i)}}(t) = r_{m+i}(t)[\frac{\dot{m}_{m+i}}{2} - \phi_{m+i+1}(t,\tau_i)]/\Delta_{m+i}(t)$$

$$i = 2,\ldots,n-m \qquad (5.2.149)$$

$$a_{12_{(m+i)}}(t) = - r_{m+i}(t)[\frac{m_{m+i}(t)}{2} + B_{m+i}n_{m+i}(t)]/2\Delta_{m+i}(t)$$

$$i = 2,\ldots,n-m \qquad (5.2.150)$$

$$a_{13_{(m+i)}}(t) = [r_{m+i}(t)m_{m+i}(t) + B_{m+i}n_{m+i}(t)(\frac{\dot{m}_{m+i}}{2} +$$

$$r_{m+i}(t) - \phi_{m+i+1}(t,\tau_i)]/2\Delta_{m+i}(t)$$

$$i = 2,\ldots,(n-m) \qquad (5.2.151)$$

where $\Delta_{m+i}(t)$ is the determinant of the matrix $\underline{B}_{m+i}(t)$. Thus (5.2.148) is rewritten as:

$$Y_{m+i-1}(t,\tau_{i-1}) = \frac{B_{m+i}n_{m+i}(t)}{2r_{m+i}(t)} q_{\xi_{m+i}}(t) \qquad (5.2.152)$$

and (5.2.146) as:

$$Q_{\xi_{m+i}}(t) = - \frac{[m_{m+i}(t) + B_{m+i}n_{m+i}(t)]}{\dot{m}_{m+i}(t) - 2\phi_{m+i+1}(t,\tau_i)} q_{\xi_{m+i}}(t) \qquad (5.2.153)$$

Substituting (5.2.143) in (5.2.140) one obtains:

$$\frac{2q_{\xi_{m+i}}(t)}{a_{11_{m+i}}(t)} + A_{m+i}(t)n_{m+i}(t) + p_{m+i+1}(t,\tau_i) = e_{m+i} \qquad (5.2.154)$$

with

$$e_{m+i} = \frac{2\eta_{m+i}}{\int_0^{T_f} a_{11_{m+i}}(t)dt} \qquad (5.2.155)$$

The expression for $a_{11_{m+i}}(t)$ from the inverse operation given by (5.2.149) after substituting for $\Delta_{m+i}(t)$ yields:

$$\frac{1}{a_{11_{m+i}}(t)} = C_{m+i}n_{m+i}(t) - \frac{[m_{m+i}(t) + B_{m+i}n_{m+i}(t)]^2}{2\dot{m}_{m+i}(t) - 4\phi_{m+i+1}(t,\tau_i)}$$

$$- \frac{B_{m+i}{}^2 n_{m+i}{}^2(t)}{4r_{m+i}(t)} \qquad (5.2.156)$$

Substituting (5.2.156) in (5.2.154) and using (5.2.152) and (5.2.153) we have

$$2C_{m+i}n_{m+i}(t)q_{\xi_{m+i}}(t) + A_{m+i}(t)n_{m+i}(t) + p_{m+i+1}(t,\tau_i)$$

$$-B_{m+i}n_{m+i}(t)\, Y_{m+i-1_\xi}(t,\tau_{i-1}) + [m_{m+i}(t) + B_{m+i}n_{m+i}(t)]$$

$$Q_{\xi_{m+i}}(t) = e_{m+i} \qquad (5.2.157)$$

Let

$$Z_{m+i}(t) = 2C_{m+i}n_{m+i}(t)q_{\xi_{m+i}}(t) + A_{m+i}(t)n_{m+i}(t)$$

$$- B_{m+i}n_{m+i}(t)\, Y_{m+i-1}(t,\tau_{i-1}) \qquad (5.2.158)$$

then differentiating (5.2.157) and substituting (5.2.158) and (5.2.57) one obtains:

$$\dot{Z}_{m+i}(t) + B_{m+i}\dot{n}_{m+i}(t)Q_{\xi_{m+i}}(t) + B_{m+i}n_{m+i}(t)q_{\xi_{m+i}}(t)$$

$$+ \frac{d}{dt}[m_{m+i}(t)Q_{\xi_{m+i}}(t)] + 2\psi_{m+i}(\tau_i,\tau_i)r_{m+i+1}(t+\tau_i) = 0$$

$$t \leq T_f - \tau_i \qquad (5.2.159)$$

and

$$\dot{Z}_{m+i}(t) + B_{m+i}\hat{n}_{m+i}(t)Q_{\xi_{m+i}}(t) + B_{m+i}n_{m+i}(t)q_{\xi_{m+i}}(t)$$

$$+ \frac{d}{dt}[m_{m+i}(t)Q_{\xi_{m+i}}(t)] = 0 \qquad t > T_f - \tau_i \qquad (5.2.160)$$

Substituting (5.2.56) in (5.2.153) and using the result in eliminating $m_{m+i}(t)$ from (5.2.159) and (5.2.160) these yield:

$$\dot{Z}_{m+i}(t) + B_{m+i}\hat{n}_{m+i}(t)Q_{m+i}(t) + 2r_{m+i+1}(t+\tau_i)$$

$$[\psi_{m+i}(\tau_i,\tau_i) + Q_{m+i}(t)] = 0 \qquad t \le T_f - \tau_i \qquad (5.2.161)$$

and

$$\dot{Z}_{m+i}(t) + B_{m+i}\hat{n}_{m+i}(t)Q_{m+i}(t) = 0 \quad t > T_f - \tau_i \qquad (5.2.162)$$

Rewriting (5.2.152) advanced one plant and with time lead τ_i as:

$$B_{m+i+1}n_{m+i+1}(t+\tau_i)q_{\xi_{m+i+1}}(t+\tau_i) = 2r_{m+i+1}(t+\tau_i)$$

$$[\psi_{m+i}(\tau_i,\tau_i) + Q_{m+i}(t)] \qquad (5.2.163)$$

one can eliminate $r_{m+i+1}(t+\tau_i)$ from (5.2.161) to obtain:

$$\dot{Z}_{m+i}(t) + B_{m+i}\hat{n}_{m+i}(t)Q_{\xi_{m+i}}(t) + B_{m+i+1}n_{m+i+1}(t+\tau_i)$$

$$q_{\xi_{m+i+1}}(t+\tau_i) = 0 \qquad t \le T_f - \tau_i \qquad (5.2.164)$$

Using (5.2.158) in (5.2.162) and (5.2.164) one obtains:

$$\frac{d}{dt}[2C_{m+i}n_{m+i}(t)\dot{Q}_{\xi_{m+i}}(t) + A_{m+i}n_{m+i}(t) - B_{m+i}n_{m+i}(t)$$

$$Y_{m+i-1}(t,\tau_{i-1})] + B_{m+i}\hat{n}_{m+i}(t)Q_{\xi_{m+i}}(t) + B_{m+i+1}n_{m+i+1}(t+\tau_i)$$

$$q_{\xi_{n+i+1}}(t+\tau_i) = 0 \qquad t \in [0,T_f - \tau_i] \qquad (5.2.165)$$

and

$$\frac{d}{dt}[2C_{m+i}n_{m+i}(t)\dot{Q}_{\xi_{m+i}}(t) + A_{m+i}(t)n_{m+i}(t) - B_{m+i}$$

$$n_{m+i}(t)\ Y_{m+i-1}(t,\tau_{i-1})] + B_{m+i}\dot{n}_{m+i}(t)Q_{m+i}(t) = 0$$

$$t\ \varepsilon(T_f - \tau_i, T_f] \qquad (5.2.166)$$

The equations (5.2.165) and (5.2.166) thus obtained depend on $n_{m+i}(t)$ and $Q_{m+i}(t)$. The multipliers $m_{m+i}(t)$ and $r_{m+i}(t)$ were eliminated as a result of invoking the corresponding constraints. This holds for $i = 2,\ldots,n\text{-}m$ or hydro-plants affected by plants which are upstream of them. Note that for $i= n\text{-}m$, the terms in r_{m+i+1} do not exist as in the cases $i = 2,\ldots,n\text{-}m\text{-}1$. However, it can be shown that (5.2.165) and (5.2.166) hold for $i = n\text{-}m$ by repeating the same analysis.

For $i = 1$, using (5.2.70), (5.2.71), (5.2.77) and (5.2.87) one obtains:

$$V_{m+1_{(1)}}(t) = [A_{m+1}(t)n_{m+1}(t) + m_{m+1}(t)]/C_{m+1}n_{m+1}(t) \qquad (5.2.167)$$

$$V_{m+1_{(2)}}(t) = -\ [\dot{m}_{m+1}(t)-\dot{p}_{m+2}(t,\tau_1)]/[\phi_{m+2}(t,\tau_1) + B_{m+1}\frac{\dot{n}_{m+1}(t)}{2}] \qquad (5.2.168)$$

Thus (5.2.138) and (5.2.139) become:

$$2C_{m+1}n_{m+1}(t)q_{\xi_{m+1}}(t) + A_{m+1}(t)n_{m+1}(t) + m_{m+1}(t) = e_{m+1} \qquad (5.2.169)$$

$$Q_{\xi_{m+1}}(t) = \frac{\dot{m}_{m+1}(t) - \dot{p}_{m+2}(t,\tau_1)}{2\phi_{m+2}(t,\tau_1) + B_{m+1}\dot{n}_{m+1}(t)} \qquad (5.2.170)$$

Differentiating (5.2.169) and substituting (5.2.170) for $\dot{m}_{m+1}(t)$ one obtains

$$\frac{d}{dt}[2C_{m+1}n_{m+1}(t)\dot{Q}_{\xi_{m+1}}(t) + A_{m+1}(t)n_{m+1}(t)]$$

$$+ B_{m+1}\dot{n}_{m+1}(t)Q_{\xi_{m+1}}(t) + B_{m+2}n_{m+2}(t+\tau_1)\dot{Q}_{\xi_{m+2}}(t+\tau_1) = 0$$

$$t \in [0,T_f - \tau_1] \qquad (5.2.171)$$

$$\frac{d}{dt}[2C_{m+1}n_{m+1}(t)\dot{Q}_{\xi_{m+1}}(t) + A_{m+1}(t)n_{m+1}(t)]$$

$$+ B_{m+1}\dot{n}_{m+1}(t)Q_{m+1_\xi}(t) = 0 \quad t \in (T_f - \tau_1, T_f] \qquad (5.2.172)$$

Note that in (5.2.171) and (5.2.172) elimination of $r_{m+2}(t+\tau_1)$ was made possible by utilizing (5.2.153) for i = 2.

We are now in a position to rewrite (5.2.165), (5.2.166), (5.2.171) and (5.2.172) in a unified fashion as:

$$\frac{d}{dt}[2C_{m+i}n_{m+i}(t)\dot{Q}_{\xi_{m+i}}(t) + A_{m+i}(t)n_{m+i}(t)]$$

$$+ B_{m+i}\dot{n}_{m+i}(t)Q_{\xi_{m+i}}(t) + g_{m+i}(t,\tau_i,\tau_{i-1}) = 0$$

$$i = 1,\ldots,n-m \qquad (5.2.173)$$

with the boundary conditions:

$$Q_{\xi_{m+i}}(o) = 0 \qquad i = 1,\ldots,n-m \qquad (5.2.174)$$

$$Q_{\xi_{m+i}}(T_f) = b_{m+i} \qquad i = 1,\ldots,n-m \qquad (5.2.175)$$

where

$$g_{m+1}(t,\tau_1) = B_{m+2}n_{m+2}(t+\tau_1)q_{\xi_{m+2}}(t+\tau_1) \qquad t\ \varepsilon[0,T_f-\tau_1]$$

$$= 0 \qquad t\ \varepsilon[T_f-\tau_1,T_f]$$

(5.2.176)

$$g_{m+i}(t,\tau_i,\tau_{i-1}) = B_{m+i+1}n_{m+i+1}(t+\tau_i)q_{\xi_{m+i+1}}(t+\tau_i)$$

$$- \frac{d}{dt}[B_{m+i}n_{m+i}(t)\psi_{m+i-1}(t,\tau_{i-1})]$$

$$t\ \varepsilon[0,\tau_{i-1}]$$

$$= B_{m+i+1}n_{m+i+1}(t+\tau_i)q_{\xi_{m+i+1}}(t+\tau_i)$$

$$- \frac{d}{dt}[B_{m+i}n_{m+i}(t)\{\psi_{m+i-1}(\tau_{i-1},\tau_{i-1})$$

$$+ Q_{\xi_{m+i-1}}(t-\tau_{i-1})\}]$$

$$t\ \varepsilon(\tau_{i-1},T_f-\tau_{i-1}]$$

$$= - \frac{d}{dt}[B_{m+i}n_{m+i}(t)\{\psi_{m+i-1}(\tau_{i-1},\tau_{i-1})$$

$$+ Q_{m+i-1}(t-\tau_{i-1})\}] \qquad t\ \varepsilon(T_f-\tau_{i-1},T_f]$$

$$i = 2,\ldots,n-m-1 \qquad (5.2.177)$$

$$g_n(t,\tau_{n-m-1}) = - \frac{d}{dt}[B_n n_n(t)\psi_{n-1}(t,\tau_{n-m-1})]$$

$$t\ \varepsilon[0,\tau_{n-m-1})$$

$$= - \frac{d}{dt}[B_n n_n(t)\{\psi_{n-1}(\tau_{n-m-1},\tau_{n-m-1})$$

$$+ Q_{\xi_{n-1}}(t-\tau_{n-m-1})\}] \qquad t\ \varepsilon(\tau_{n-m-1},T_f]$$

(5.2.178)

It is interesting to note that (5.2.173) describes the optimal interaction of the volume of water discharged $Q_{\xi_{m+i}}(t)$. The solution of this equation will depend on $n_{m+i}(t)$ which is to be determined such that (5.2.28) is satisfied. The value of $n_{m+i}(t)$ depends solely on the (m+i)th plant characteristics. Moreover, the solution will also depend on $g_{m+i}(t,\tau_i,\tau_{i-1})$. This depends on the behavior of the plants immediately upstream and downstream from the (m+i)th plant.

The equations (5.2.137) and (5.2.173) will be referred to as the modified optimal solution. It can be shown that this solution satisfies the necessary and sufficient condition for optimality for control problems with time delayed arguments as given by D.K. Hughes in [12].

5.2.5 Special Cases of the Problem

It is the purpose of this subsection to show the optimal solution for some specific cases of the problem considered in this section. First we give the solution for the case when the water flow time delays between hydro stations on the same stream are negligible. The results obtained here are identical with those obtained in [7]. It is worth mentioning that this special case was formulated and solved without introducing the pseudo-control $x_{m+i}(t)$. This is possible since $x_{m+i}(t)$ was introduced to facilitate dealing with the time delays. The results obtained both ways are identical. Next, the solution when tail-race variations are neglected is given. Finally, the optimal solution for plants not on the same stream is obtained. It is shown that the solution here is identical with the results of Chapter 4. In all these cases we are dealing with the hydraulic discharge equation (5.2.173).

A. Negligible Time Delay Case

In this case $\tau_i = 0$ so that (5.2.31) yields

$$\psi_{m+i-1}(t,0) = \int_0^t q_{m+i-1}(\sigma)d\sigma \qquad t \leq 0$$

and

$$\psi_{m+i-1}(0,0) = 0 \tag{5.2.179}$$

Also (5.2.176), (5.2.177) and (5.2.178) transform into:

$$g_{m+1}(t,0) = B_{m+2}n_{m+2}(t)q_{\xi_{m+2}}(t) \tag{5.2.180}$$

$$g_{m+i}(t,0,0) = B_{m+i+1}n_{m+i+1}(t)q_{\xi_{m+i+1}}(t) - \frac{d}{dt}[B_{m+i}n_{m+i}(t)Q_{\xi_{m+i-1}}(t)]$$

$$i = 2,\ldots,(n\text{-}m\text{-}1) \tag{5.2.181}$$

$$g_n(t,0) = -\frac{d}{dt}[B_n n_n(t)Q_{\xi_{n-1}}(t)] \tag{5.2.182}$$

Thus the optimal discharges are given by (5.2.173) using (5.2.180) through (5.2.182).

B. Negligible Effect due to Tail-race Elevation

In this case $C_{m+i} = 0$, since $\beta_{T_{m+i}} = 0$. Thus (5.2.173) is rewritten for this case as:

$$Q_{\xi_{m+i}}(t) = -\frac{1}{B_{m+i}\dot{n}_{m+i}(t)}[g_{m+i}(t,\tau_i,\tau_{i-1}) + \frac{d}{dt}(A_{m+i}(t)n_{m+i}(t))] \qquad i = 1,\ldots,n\text{-}m \tag{5.2.183}$$

where the g_{m+i}'s are given by (5.2.176) through (5.2.177) in the case when time delays are present or (5.2.180) through (5.2.182) when time delays

are negligible.

C. Hydro-plants not on the same stream

In this case two observations are made. The first one is that in (5.2.7) the coupling term $q_{m+i-1}(t-\tau_{i-1})$ will be absent. The second observation is that one can assume that the τ_i's approach infinity, so that the discharge from one hydro-plant would not effect the others. These two observations can be stated as:

$$\lim_{\tau_i \to \infty} q_{m+i}(t-\tau_i) = 0 \qquad i = 1,\ldots,(n-m) \tag{5.2.184}$$

This leads to:

$$\lim_{\tau_i \to \infty} Y_{m+i}(t,\tau_i) = 0 \qquad i = 1,\ldots,(n-m) \tag{5.2.185}$$

and

$$\lim_{\tau_i \to \infty} \psi_{m+i}(t,\tau_i) = 0 \qquad i = 1,\ldots,(n-m) \tag{5.2.186}$$

The last two equations result from applying (5.2.184) to (5.2.16) and (5.2.32) respectively. Applying the last results to (5.2.176) through (5.2.178) one obtains

$$\lim_{\tau_i \to \infty} g_{m+i}(t,\tau_i,\tau_{i-1}) = 0 \quad i = 1,\ldots,n-m \tag{5.2.187}$$

Thus (5.2.173) reduces to

$$\frac{d}{dt}[2C_{m+i}n_{m+i}(t)\dot{Q}_{\xi_{m+i}}(t) + A_{m+i}(t)n_{m+i}(t)]$$

$$+ B_{m+i}\dot{n}_{m+i}(t)Q_{\xi_{m+i}}(t) = 0 \quad i = 1,\ldots,n-m \tag{5.2.188}$$

Furthermore, if tail-race elevation is neglected, then (5.2.188) reduces to:

$$Q^{(5)}_{\xi_{m+i}}(t) = - \frac{1}{B_{m+i}\dot{n}_{m+i}(t)} \frac{d}{dt}[A_{m+i}(t)n_{m+i}(t)]$$

$$i = 1,\ldots,n-m \qquad (5.2.189)$$

It is our intention now to show that (5.2.189) is identical with (4.4.97). To this end we rewrite (4.4.97) as:

$$Q^{(4)}_{\xi_{m+i}}(t) = \frac{r_{m+i}(t)}{E_{m+i}\dot{n}_{m+i}(t)} \qquad (5.2.190)$$

and using (4.4.20) this is given by:

$$Q^{(4)}_{\xi_{m+i}}(t) = \frac{1}{E_{m+i}}[N_{m+i}(t) - \frac{n_{m+i}(t)}{\dot{n}_{m+i}(t)} \dot{N}_{m+i}(t)] \qquad (5.2.191)$$

Also by using (4.4.10) and (4.4.11), (5.2.190) reduces to:

$$Q^{(4)}_{\xi_{m+i}}(t) = S_{i_{m+i}} h_{i_{m+i}}(o) + \int_o^t i_{m+i}(\sigma)d\sigma - \frac{n_{m+i}(t)}{\dot{n}_{m+i}(t)} i_{m+i}(t)$$

Note that S_i here is that of Chapter 4, which means that $S_{i_{m+i}} h_{i_{m+i}}(o)$ is in fact the initial storage $S_{m+i}(o)$ of this chapter. Thus we have

$$Q^{(4)}_{\xi_{m+i}}(t) = [S_{m+i}(o) + I_{m+i}(t)] - \frac{n_{m+i}(t)}{\dot{n}_{m+i}(t)} i_{m+i}(t) \qquad (5.2.192)$$

Using (5.2.15) this is given by

$$Q^{(4)}_{\xi_{m+i}}(t) = D_{m+i}(t) - \frac{n_{m+i}(t)}{\dot{n}_{m+i}(t)} \dot{D}_{m+i}(t) \qquad (5.2.193)$$

Now (5.2.189) is rewritten using (5.2.24) and (5.2.25) as:

$$Q^{(5)}_{\xi_{m-i}}(t) = \frac{\alpha_{m+i}}{\beta_{y_{m+i}}} + D_{m+i}(t) - \frac{n_{m+i}(t)}{\dot{n}_{m+i}(t)} \dot{D}_{m+i}(t) \qquad (5.2.194)$$

Observe that in Chapter 4, α_{m+i} was taken as zero. Thus (5.2.194) and (5.2.193) are identical. This proves that the results of section 4.4 are a special case of the results obtained in this section.

5.2.6 Implementing the Optimal Solution

The modified optimal solution obtained in (5.2.4) contains the unknown multiplier functions $n_i(t)$ and $\lambda(t)$. These are to be determined such that the constraints (5.2.10), (5.2.27) and (5.2.28) are satisfied. In this subsection, the reduction of the equations specifying the optimal schedules for the general system is given.

Let the (nxn) symmetric matrix:

$$\underline{B}_p^{-1}(t) = (b_{ij_p}(t)) \tag{5.2.195}$$

be the inverse of $\underline{B}_p(t)$ as given by (5.2.76). Note that elements of $\underline{B}_p(t)$ are dependent on $\lambda(t)$ so that one may write

$$\underline{B}_p^{-1}(t) = (b_{ij_p}(\lambda(t))) \tag{5.2.196}$$

From (5.2.64) and (5.2.196), the expression for $\underline{V}_p(t)$ given by (5.2.133) and (5.2.87) in the component form (5.2.134) is:

$$V_{p_i}(t) = \sum_{j=1}^{m} [\beta_j - \lambda(t)(1 - B_{jo})]b_{ji_p}(\lambda(t))$$
$$+ \sum_{j=m+1}^{n} [n_j(t) - \lambda(t)(1 - B_{jo})b_{ji_p}(\lambda(t)) \tag{5.2.197}$$

and the optimal power generations given by (5.2.137) component-wise are:

$$P_{s_{i_\xi}}(t) = - V_{p_i}(t)/2 \qquad i = 1,\dots,m \tag{5.2.198}$$

$$P_{h_{i_\xi}}(t) = - V_{p_i}(t)/2 \qquad i = m+1,\dots,n \tag{5.2.199}$$

Here it is noted that the power generations depend on the $n_i(t)$ and $\lambda(t)$.

The optimal power generations should satisfy (5.2.10), thus the following relation must hold true:

$$P_D(t) + \frac{1}{2}\sum_{i=1}^{n} V_{p_i}(t) + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} V_{p_i}(t)B_{ij}V_{p_j}(t)$$

$$- \frac{1}{2}\sum_{i=1}^{n} B_{io}V_{p_i}(t) + K_{Lo} = 0 \qquad (5.2.200)$$

This is an algebraic equation in $\lambda(t)$ and the (n-m) functions $n_i(t)$.

The optimal hydro-generations should satisfy (5.2.27) and (5.2.28), thus

$$V_{p_{m+1}}(t) = 2[A_{m+1}(t)\dot{Q}_{m+1}(t) + B_{m+1}\dot{Q}_{m+1}(t)Q_{m+1}(t)$$

$$+ C_{m+1}\dot{Q}_{m+1}^{\,2}(t)] \qquad (5.2.201)$$

$$V_{p_{m+i}}(t) = 2[A_{m+i}(t)\dot{Q}_{m+i}(t) + B_{m+i}\dot{Q}_{m+i}(t)Q_{m+i}(t)$$

$$- B_{m+i}\dot{Q}_{m+i}(t)\, Y_{m+i-1}(t,\tau_{i-1})$$

$$+ C_{m+i}\dot{Q}_{m+i}^{\,2}(t)] \qquad i = 2,\ldots,n-m \quad (5.2.202)$$

Here $Y_{m+i-1}(t,\tau_{i-1})$ are given by (5.2.32).

The problem of implementing the optimal solution is thus reduced to that of finding [2(n-m)+1] unknown functions $Q_i(t)[i=m+1,\ldots,n]$, $n_i(t)[i=m+1,\ldots,n]$ and $\lambda(t)$. These are obtained by solving simultaneously (5.2.173), (5.2.200), (5.2.201) and (5.2.202). These are exactly [2(n-m)+1] equations in the unknown functions. Due to the nonlinearity of these equations, one inevitably resorts to iterative techniques.

Consider equation (5.2.173), rewritten as

$$\ddot{Q}_{\xi_{m+i}}(t) + \frac{\dot{n}_{m+i}(t)}{n_{m+i}(t)}\dot{Q}_{\xi_{m+i}}(t) + \frac{B_{m+i}}{2C_{m+i}}\frac{\dot{n}_{m+i}(t)}{n_{m+i}(t)}Q_{\xi_{m+i}}(t)$$

$$+ g_{m+i}(t,\tau_i,\tau_{i-1}) + \frac{\dot{A}_{m+i}(t)}{2C_{m+i}} + \frac{A_{m+i}(t)}{2C_{m+i}}\frac{\dot{n}_{m+i}(t)}{n_{m+i}(t)} = 0$$

$$i = 1,\ldots,n-m \qquad (5.2.203)$$

Let

$$\rho_{m+i}(t) = \frac{\dot{n}_{m+i}(t)}{n_{m+i}(t)} \qquad i = 1,\ldots,n-m \qquad (5.2.204)$$

$$\varepsilon_{m+i} = \frac{B_{m+i}}{2C_{m+i}} \qquad i = 1,\ldots,n-m \qquad (5.2.205)$$

$$G_{m+i}(t,\tau_i,\tau_{i-1},n_{i_{m+i}}(t),\dot{n}_{i_{m+i}}(t)) = g_{m+i}(t,\tau_i,\tau_{i-1})$$

$$+ \frac{\dot{A}_{m+i}(t)}{2C_{m+i}} + \frac{A_{m+i}(t)}{2C_{m+i}}\rho_{m+i}(t) \qquad i = 1,\ldots,n-m \qquad (5.2.206)$$

Then (5.2.203) reduces to:

$$\ddot{Q}_{\xi_{m+i}}(t) + \rho_{m+i}(t)\dot{Q}_{\xi_{m+i}}(t) + \varepsilon_{m+i}\rho_{m+i}(t)Q_{\xi_{m+i}}(t)$$

$$+ G_{m+i} = 0 \qquad i = 1,\ldots,(n-m) \qquad (5.2.207)$$

Furthermore, let

$$\underline{Z}_{m+i}(t) = \text{col.}[Z^{(1)}_{m+i}(t), Z^{(2)}_{m+i}(t)]$$

where

$$Z^{(1)}_{m+i}(t) = Q_{\xi_{m+i}}(t) \text{ and } Z^{(2)}_{m+i}(t) = \dot{Q}_{\xi_{m+i}}(t)$$

$$i = 1,\ldots,(n-m) \qquad (5.2.208)$$

Then (5.2.207) reduces to the linear vector equation:

$$\dot{\underline{Z}}_{m+i}(t) = \underline{R}_{m+i}(t)\underline{Z}_{m+i}(t) + \underline{F}_{m+i}(t)$$

$$i = 1,\ldots,(n-m) \qquad (5.2.209)$$

with

$$\underline{R}_{m+i}(t) = \begin{bmatrix} 0 & 1 \\ -\varepsilon_{m+i}\rho_{m+i}(t) & -\rho_{m+i}(t) \end{bmatrix}$$

$$i = 1,\ldots,(n-m)$$

$$\underline{F}_{m+i}(t) = \text{col.}[0, -G_{m+i}] \qquad i = 1,\ldots,(n-m) \qquad (5.2.210)$$

Here the boundary conditions $Q_{\xi_{m+i}}(o) = 0$ and $Q_{\xi_{m+i}}(T_f) = b_{m+i}$ can be written as:

$$\underline{M}\,\underline{Z}_{m+i}(o) + \underline{N}\,\underline{Z}_{m+i}(T_f) = \underline{C}'_{m+i}$$

$$i = 1,\ldots,n-m \qquad (5.2.211)$$

with

$$\underline{M} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \underline{N} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad (5.2.212)$$

$$\underline{C}'_{m+i} = \text{col.}[0, b_{m+i}] \qquad i = 1,\ldots,n-m \qquad (5.2.213)$$

Equation (5.2.209) with the boundary condition (5.2.211)are next transformed into an integral form. The results obtained here rely heavily on those given by P.L. Falb and J.L. De Jong in [56]. The following is an integral representation of (5.2.209) and (5.2.211):

$$Z^{(1)}_{m+i}(t) = \frac{b_{m+i}t}{T_f} + \int_o^t \frac{s[T_f-t]}{T_f} f_{m+i}(s)ds + \int_t^{T_f} \frac{t[T_f-s]}{T_f} f_{m+i}(s)ds$$

$$i = 1,\ldots,(n-m) \qquad (5.2.214)$$

$$Z_{m+i}^{(2)}(t) = \frac{b_{m+i}}{T_f} + \int_0^t - \frac{s}{T_f} f_{m+i}(s)ds + \int_t^{T_f} \frac{T_f - s}{T_f} f_{m+i}(s)ds$$

$$i = 1,\ldots,(n-m) \qquad (5.2.215)$$

Here

$$f_{m+i}(s) = \varepsilon_{m+i}\rho_{m+i}(s)Z_{m+i}^{(1)}(s) + \rho_{m+i}(s)Z_{(m+i)}^{(2)}(s)$$

$$+ G_{m+i}(s) \qquad i = 1,\ldots,(n-m) \qquad (5.2.216)$$

Note that in (5.2.215), satisfying the boundary condition (5.2.211) is guaranteed during the search for the required solution. For the sake of simplicity, the practical application of the suggested computational scheme is shown by way of an example. This example is concerned with a one-thermal, two hydro-plants on the same stream system. This is the subject of the next subsection.

5.2.7 Practical Application

A computer program was written to solve (5.2.200), (5.2.201), (5.2.202), (5.2.214) and (5.2.215) to obtain optimum generation schedules for a sample system. The sample system is characterized by a diagonal loss matrix (i.e. $B_{ij} = 0$, $i \neq j$). The number of hydro-plants is two and they are on the same stream. There is only one thermal plant in the system. For this system, the equations describing the optimum mode are:

$$Z_2^{(1)}(t) = \frac{1}{T_f}[b_2 t + \int_0^t s[T_f - t]f_2(s)ds + \int_t^{T_f} t[T_f - s]f_2(s)ds] \qquad (5.2.217)$$

$$Z_2^{(2)}(t) = \frac{1}{T_f}[b_2 + \int_0^t -sf_2(s)ds + \int_t^{T_f} (T_f - s)f_2(s)ds] \qquad (5.2.218)$$

$$Z_3^{(1)}(t) = \frac{1}{T_f}[b_3 t + \int_0^t s[T_f-t]f_3(s)ds + \int_t^{T_f} t[T_f-s]f_3(s)ds] \tag{5.2.219}$$

$$Z_3^{(2)}(t) = \frac{1}{T_f}[b_3 + \int_0^t - sf_3(s)ds + \int_t^{T_f} (T_f-s)f_3(s)ds] \tag{5.2.220}$$

Define

$$\underline{Y}(t) = \text{col.}[y_1(t), y_2(t), y_3(t), y_4(t)] \tag{5.2.221}$$

where

$$y_1(t) = Z_2^{(1)}(t) \tag{5.2.222}$$

$$y_2(t) = Z_2^{(2)}(t) \tag{5.2.223}$$

$$y_3(t) = Z_3^{(1)}(t) \tag{5.2.224}$$

$$y_4(t) = Z_3^{(2)}(t) \tag{5.2.225}$$

Then equations (5.2.217) through (5.2.220) define the operator equation

$$\underline{Y}(t) = T(\underline{Y}(t)) \tag{5.2.226}$$

This equation can be solved iteratively using a modified contraction mapping algorithm of the form:

$$\underline{Y}^{(n+1)} = [\underline{I} - \underline{U}]^{-1}[\underline{T}(\underline{Y}^{(n)}) - \underline{U}(\underline{Y}^{(n)})] \tag{5.2.227}$$

Here $\underline{I}$ is the identity operator, $\underline{U}$ is any operator which is linear and $[\underline{I}-\underline{U}]$ is invertible. The convergence conditions for this algorithm are given in Appendix A.

The initial estimate of the unknown variables is taken as:

$$y_1^{(o)}(t) = b_2 t/T_f \tag{5.2.228}$$

$$y_2^{(o)}(t) = b_2/T_f \tag{5.2.229}$$

TABLE 5.1

SAMPLE SYSTEM CHARACTERISTICS

Thermal Plant:

$\beta_1 = 4.0$ $\qquad$ $\gamma_1 = 0.12 \times 10^{-2}$

Loss Formula Coefficients:

$B_{11} = 0.16 \times 10^{-3}$ $\qquad$ $B_{22} = 0.22 \times 10^{-3}$ $\qquad$ $B_{33} = 0.16 \times 10^{-3}$

The Hydro Plants:

$\eta_2 = 0.708$	$\eta_3 = 0.708$
$S_2(o) = 0.72 \times 10^{13}$	$S_3(o) = 0.144 \times 10^{13}$
$\alpha_2 = 0.0$	$\alpha_3 = 0.0$
$\beta_{T_2} = 0.54 \times 10^{-6}$	$\beta_{T_3} = 0.2047 \times 10^{-6}$
$\beta_{y_2} = 0.1389 \times 10^{-10}$	$\beta_{y_3} = 0.1389 \times 10^{-9}$
$i_2(t) = 0.1 \times 10^{6}$	$i_3(t) = 0.1 \times 10^{7}$

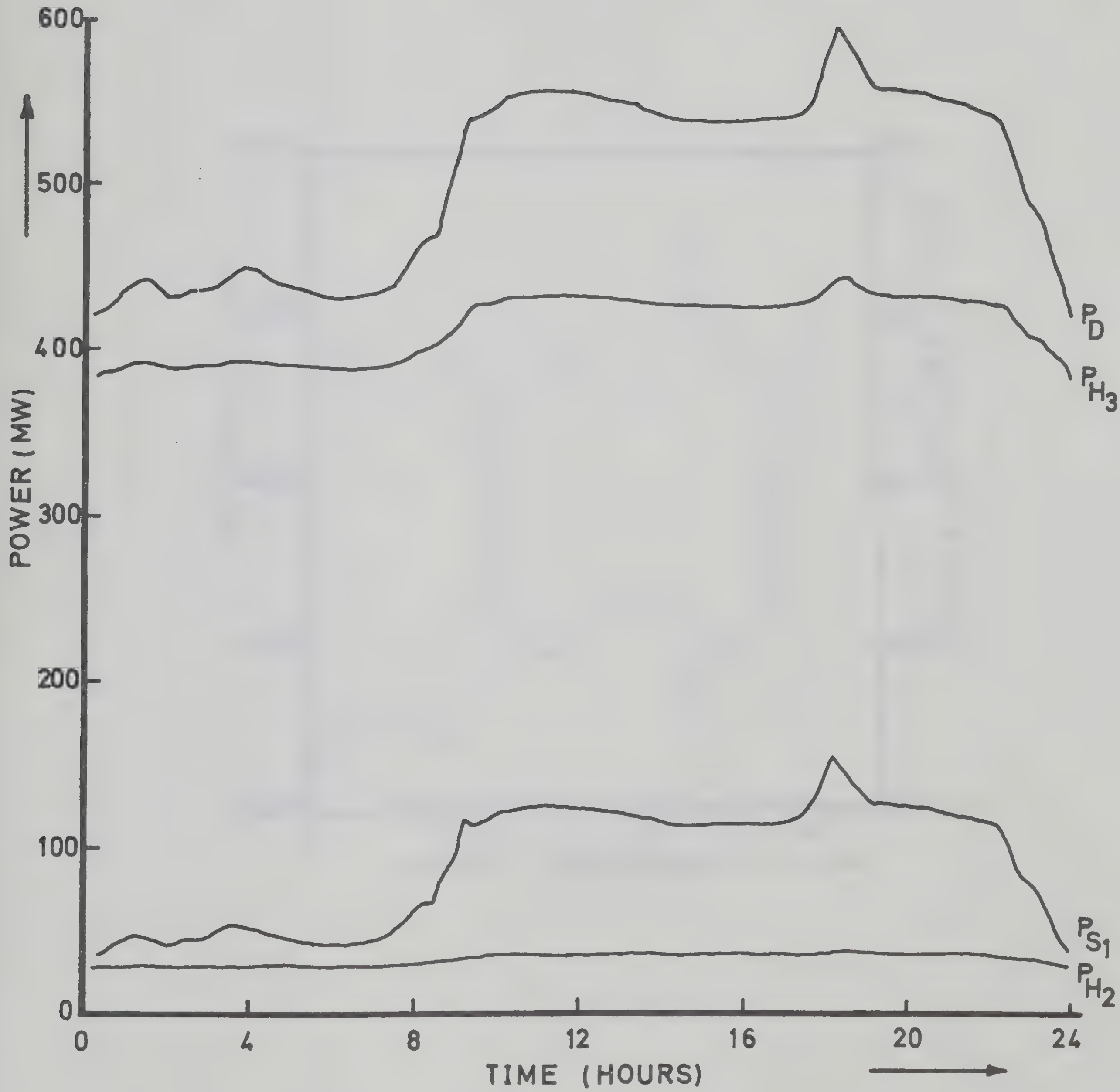


Figure 5.2 Optimum Schedules for the Sample System.

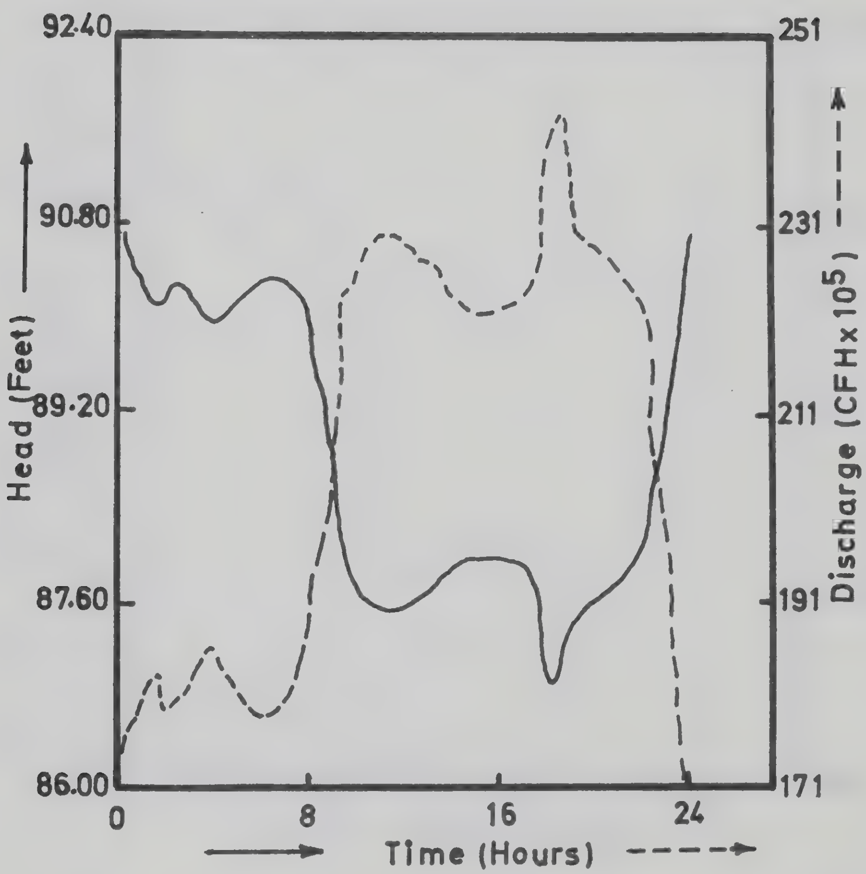


Figure 5.3 Optimum Head Variations and Rate of Water Discharge for the Upstream Plant

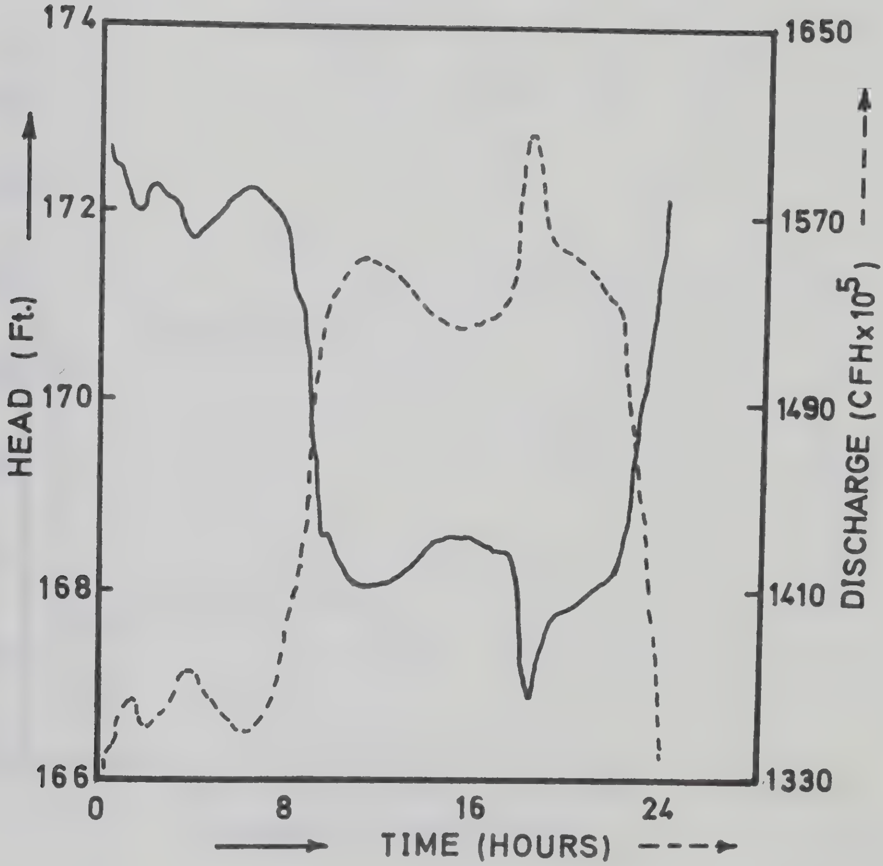


Figure 5.4 Optimum Head Variations and Rate of Water Discharge for the Downstream Plant.

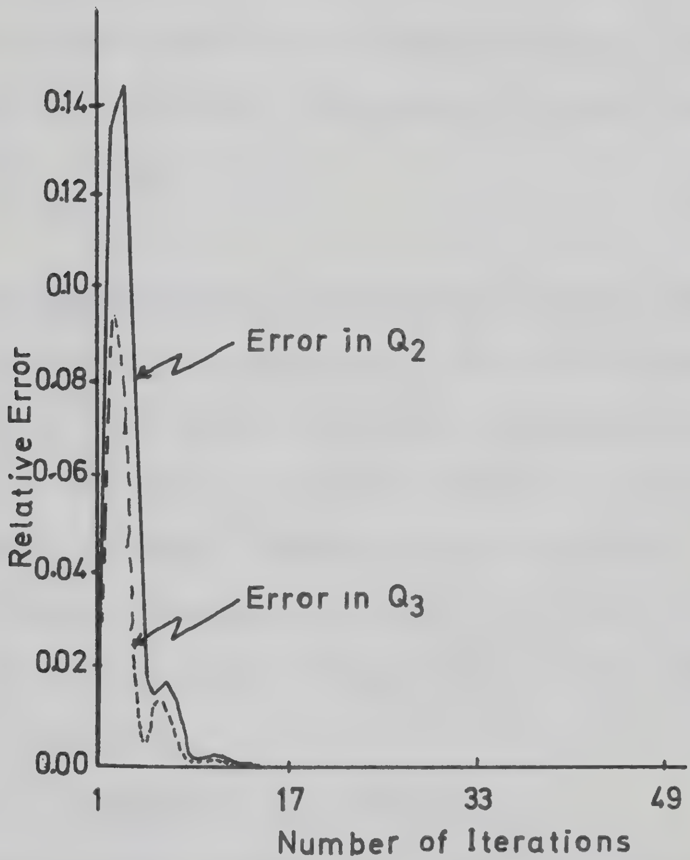


Figure 5.5 Variation of Relative Error with Number of Iterations.

$$y_3^{(o)}(t) = b_3 t/T_f \tag{5.2.230}$$

$$y_4^{(o)}(t) = b_3/T_f \tag{5.2.231}$$

A sample system whose particulars are summarized in Table (5.1) was used to test the method. Optimum loading schedules obtained are shown in Figures (5.2), (5.3) and (5.4). Figure (5.5) shows the variation of the relative error between successive approximations with the number of iterations. The operator $\underline{U}$ for this test problem was taken as $\underline{U} = -0.95\underline{I}$.

5.3 Power System with Multiple Chains of Variable Head Hydro-plants

This section is concerned with the case of a power system with series plants (on the same stream), multiple chains of plants and imtermediate reservoirs. The variety of models that can be considered from a theoretical standpoint is infinite. However, a practical model is chosen in this section. Here the formulation adopted is applicable to any practical system with a larger number of hydro-plants. The results obtained here are reported in [57].

5.3.1 Statement of the Problem

A hydro-thermal electric power system is considered. The system has one thermal plant and eight hydro-plants. The hydraulic portion of the system is shown in Fig. (5.6). The operating conditions require minimizing.

$$J_o = \int_0^{T_f} [\alpha + \beta P_{s_g}(t) + \gamma P_{s_g}^2(t)]dt \tag{5.3.1}$$

while satisfying the following constraints:

1. The power balance equation.

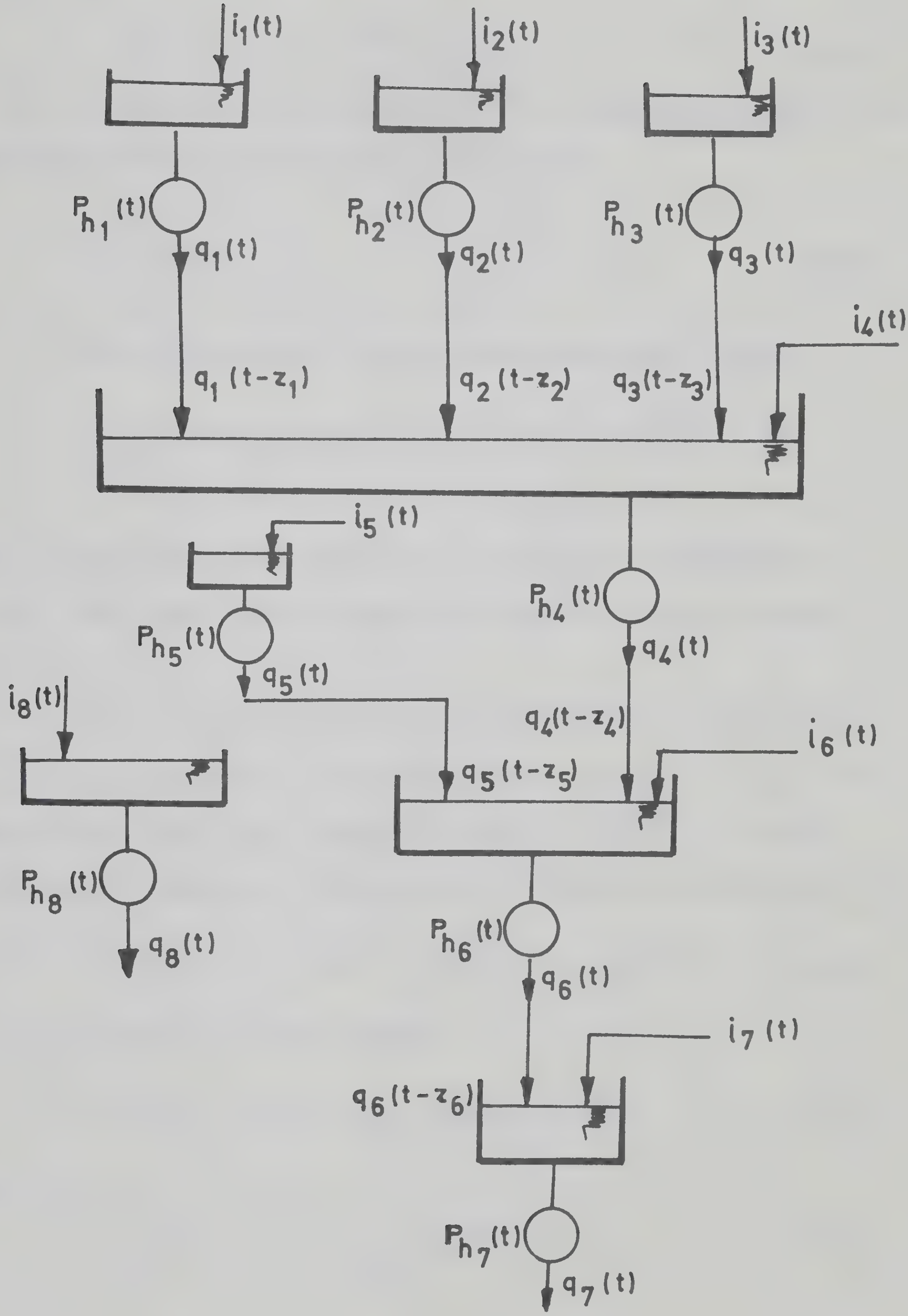


Figure 5.6 General Layout of the Hydro-Plants for the System in Section 5.3.

$$P_D(t) = P_{s_g}(t) + \sum_{i=1}^{8} P_{h_i}(t) - \sum_{i=1}^{9}\sum_{j=1}^{9} P_i(t)B_{ij}P_j(t)$$

$$- \sum_{i=1}^{9} B_{io}P_i(t) \qquad (5.3.2)$$

2. The volume of water discharged at the hydro-plants during the optimization interval is a prespecified constant:

$$\int_0^{T_f} q_i(\sigma)d\sigma = b_i \qquad i = 1,\dots,8 \qquad (5.3.3)$$

3. The upstream plants' active power generations satisfy:

$$P_{h_i}(t) + A_i(t)q_i(t) + B_iq_i(t)Q_i(t) + C_iq_i^2(t) = 0$$

$$i = 1,2,3,5,8 \qquad (5.3.4)$$

These are upstream plants so that all of them can be represented by (5.2.23). Note that $A_i(t)$, B_i and C_i are given by (5.2.24) through (5.2.26).

4. For the intermediate plants; the situation is illustrated in Fig. (5.7). Let there be (i-k) plants upstream from the (i+1)st. The flow from each plant has a transport delay of $\tau_j (j=k,\dots,i)$ to the (i+1)st plant. Then the (i+1)st reservoir's dynamics are expressed by:

$$\dot{S}_{i+1}(t) = i_{i+1}(t) + \sum_{j=k}^{i} q_j(t-\tau_j) - q_{i+1}(t) \qquad (5.3.5)$$

Integrating (5.3.5) one obtains

$$S_{i+1}(t) = D_{i+1}(t) + \sum_{j=k}^{i} x_j(t) - Q_{i+1}(t) \qquad (5.3.6)$$

where

$$D_{i+1}(t) = S_{i+1}(o) + \int_0^t i_{i+1}(\sigma)d\sigma \qquad (5.3.7)$$

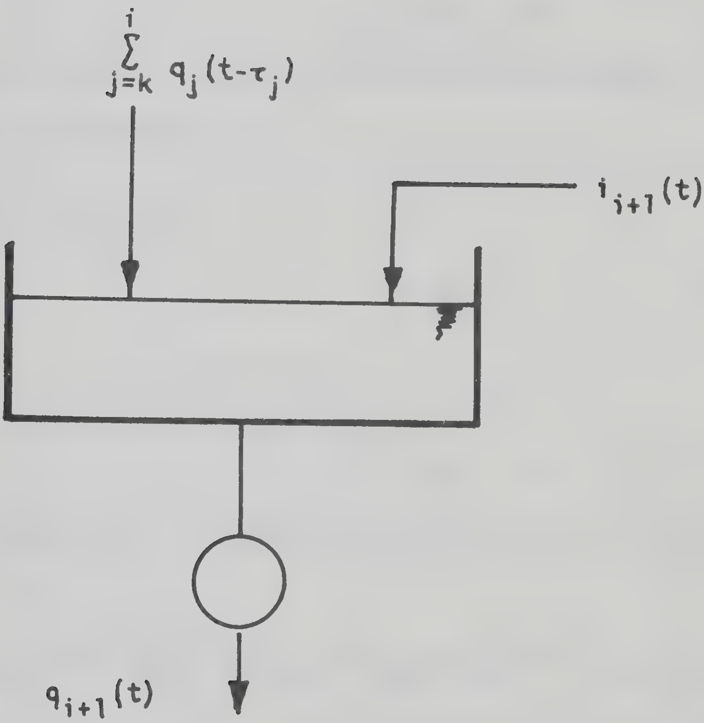


Figure 5.7 The ith Hydro-Plant's Reservoir.

$$Q_i(t) = \int_0^t q_i(\sigma)d\sigma \tag{5.3.8}$$

$$x_j(t) = \int_0^t q_j(\sigma-\tau_j)d\sigma \qquad j = k,\ldots,i \tag{5.3.9}$$

Let

$$\psi_j(t,\tau_j) = \int_{-\tau_j}^{t-\tau_j} q_j(s)ds \qquad t \leq \tau_j$$

$$j = k,\ldots,i \tag{5.3.10}$$

which is a known function of time from the previous history of the system. Then (5.3.9) reduces to:

$$x_j(t) = \psi_j(t,\tau_j) \qquad t \leq \tau_j$$

$$i = k,\ldots,i \tag{5.3.11}$$

$$= \psi_j(\tau_j,\tau_j) + Q_j(t-\tau_j) \quad t > \tau_j$$

$$j = k,\ldots,i \tag{5.3.12}$$

The hydro-power generated at the (i+1)<u>st</u> plant is given by:

$$P_{h_{i+1}}(t) = \frac{q_{i+1}(t)}{G_{i+1}}[\alpha_{i+1} + \beta_{y_{i+1}} S_{i+1}(t) - \beta_{T_{i+1}} q_{i+1}(t)] \tag{5.3.13}$$

Substituting (5.3.6) in (5.3.13) for $S_{i+1}(t)$, one obtains

$$P_{h_{i+1}}(t) + A_{i+1}(t)q_{i+1}(t) - B_{i+1}q_{i+1}(t) \sum_{j=k}^{i} x_j(t)$$

$$+ C_{i+1}q_{i+1}{}^2(t) + B_{i+1}q_{i+1}(t)Q_{i+1}(t) = 0 \tag{5.3.14}$$

In the system at hand, plant number 4 is downstream from plants number 1, 2 and 3. Also plant number 6 is downstream from 4 and 5, and 7 is downstream from the 6<u>th</u> plant. Thus applying (5.3.14) we have:

$$P_{h_4}(t) + A_4(t)q_4(t) - B_4q_4(t)[x_1(t) + x_2(t) + x_3(t)]$$

$$+ B_4q_4(t)Q_4(t) + C_4q_4^2(t) = 0 \tag{5.3.15}$$

$$P_{h_6}(t) + A_6(t)q_6(t) - B_6q_6(t)[x_4(t) + x_5(t)]$$

$$+ B_6q_6(t)Q_6(t) + C_6q_6^2(t) = 0 \tag{5.3.16}$$

$$P_{h_7}(t) + A_7(t)q_7(t) - B_7q_7(t)[x_6(t)] + C_7q_7^2(t)$$

$$+ B_7q_7(t)Q_7(t) = 0 \tag{5.3.17}$$

5.3.2 A Minimum Norm Formulation

The cost functional to be minimized is:

$$J_o = \int_0^{T_f} [\alpha + \beta P_{s_g}(t) + \gamma P_{s_g}{}^2(t)]dt \tag{5.3.18}$$

subject to satisfying (5.3.2), (5.3.3), (5.3.4), (5.3.8), (5.3.11), (5.3.12) (5.3.15), (5.3.16) and (5.3.17). Including all these constraints except for (5.3.3), (5.3.8), (5.3.11) and (5.3.12), we have to minimize

$$J_1 = \int_0^{T_f} [\{\beta - \lambda(t)(1 - B_{go})\}P_{s_g}(t) + \sum_{i=1}^{8} \{n_i(t) - \lambda(t)(1 - B_{io})\}$$

$$P_{h_i}(t) + \sum_{i=1}^{8} n_i(t)A_i(t)q_i(t) + \gamma P_{s_g}{}^2(t) + \lambda(t)$$

$$\sum_{i=1}^{9} \sum_{j=1}^{9} P_i(t)B_{ij}P_j(t) + \sum_{i=1}^{8} C_i n_i(t)q_i{}^2(t)$$

$$+ \sum_{i=1}^{8} B_i n_i(t)\dot{Q}_i(t)Q_i(t) - B_4 n_4(t)q_4(t)$$

$$[x_1(t) + x_2(t) + x_3(t)] - B_6 n_6(t)q_6(t)$$

$$[x_4(t) + x_5(t)] - B_7 n_7(t)q_7(t)x_6(t)]dt \quad (5.3.19)$$

subject to (5.3.3), (5.3.8), (5.3.11) and (5.3.12).

Consider (5.3.8), the equivalent of which is:

$$q_i(t) = \dot{Q}_i(t) \qquad i = 1,\ldots,8 \qquad (5.3.20)$$

and the equivalents of (5.3.11) and (5.3.12) are

$$x_j^{\,2}(t) = \psi_j^{\,2}(t,\tau_j) \qquad t \leq \tau_j \qquad (5.3.21)$$

$$x_j^{\,2}(t) = \psi_j^{\,2}(\tau_j,\tau_j) + 2\psi_j(\tau_j,\tau_j)Q_j(t-\tau_j)$$

$$+ Q_j^{\,2}(t-\tau_j) \qquad t \geq \tau_j \qquad (5.3.22)$$

The extra functional to be added to (5.3.19) is (so that (5.3.20) and (5.3.21) are taken care of):

$$J_{x,q,Q} = \int_0^{T_f} \sum_{i=1}^{8} [m_i(t)q_i(t) + \dot{m}_i(t)Q_i(t)]dt$$

$$+ \int_0^{T_f} \sum_{i=1}^{6} r_i(t)x_i^{\,2}(t)dt - \sum_{i=1}^{6} \int_{\tau_i}^{T_f} r_i(t)$$

$$[Q_i^{\,2}(t-\tau_i) + 2\psi_i(\tau_i,\tau_i)Q_i(t-\tau_i)]dt$$

$$(5.3.23)$$

Here terms explicitly independent of $q_i(t)$, $Q_i(t)$ and $x_i(t)$ were neglected. The functional given by (5.3.23) can further be reduced to

$$J_{x,q,Q} = \int_0^{T_f} \{ \sum_{i=1}^{8} [m_i(t)q_i(t) + \dot{m}_i(t)Q_i(t)] + \sum_{i=1}^{6}$$

$$r_i(t)x_i^2(t)]dt - \sum_{i=1}^{6} \int_0^{T_f-\tau_i} r_i(s+\tau_i)$$

$$[Q_i^2(s) + 2\psi_i(\tau_i,\tau_i)Q_i(s)]ds \qquad (5.3.24)$$

Let

$$\begin{aligned} \dot{p}_i(t,\tau_i) &= 2\psi_i(\tau_i,\tau_i)r_i(t+\tau_i) & 0 \le t \le T_f-\tau_i \\ &= 0 & T_f-\tau_i < t \le T_f \\ & & i = 1,\ldots,6 \end{aligned} \qquad (5.3.25)$$

$$\begin{aligned} \theta_i(t,\tau_i) &= r_i(t+\tau_i) & 0 \le t \le T_f-\tau_i \\ &= 0 & T_f-\tau_i < t \le T_f \\ & & i = 1,\ldots,6 \end{aligned} \qquad (5.3.26)$$

then the augmented cost functional given by J_1 of (5.3.19) plus $J_{x,q,Q}$ of (5.3.24) is given by:

$$J_2(.) = \int_0^{T_f} [(\beta - \lambda(t)(1 - B_{go}))P_{s_g}(t)$$

$$+ \sum_{i=1}^{8} [n_i(t) - \lambda(t)(1 - B_{io})]P_{h_i}(t)$$

$$+ \sum_{i=1}^{6} [n_i(t)A_i(t) + m_i(t) + p_i(t,\tau_i)]q_i(t)$$

$$+ \sum_{i=7}^{8} [n_i(t)A_i(t) + m_i(t)]q_i(t)$$

$$+ \sum_{i=1}^{8} \dot{m}_i(t)Q_i(t) + \gamma P_{s_g}{}^2(t)$$

$$+ \lambda(t) \sum_{i=1}^{9} \sum_{j=1}^{9} P_i(t)B_{ij}P_j(t)$$

$$+ \sum_{i=1}^{8} C_i n_i(t) q_i{}^2(t) + \sum_{i=1}^{8} - \frac{B_i \dot{n}_i(t)}{2} Q_i{}^2(t)$$

$$- B_4 n_4(t) q_4(t)[x_1(t) + x_2(t) + x_3(t)]$$

$$- B_6 n_6(t) q_6(t)[x_4(t) + x_5(t)] - B_7 n_7(t) q_7(t) x_6(t)$$

$$+ \sum_{i=1}^{6} r_i(t) x_i{}^2(t) + \sum_{i=1}^{6} - \theta_i(t,\tau_i) Q_i{}^2(t)]dt \qquad (5.3.27)$$

Define the control vector by:

$$\underline{u}(t) = \text{col.}[\underline{P}(t), \underline{W}_1(t), \underline{W}_2(t), \underline{W}_3(t), \ldots, \underline{W}_8(t)] \qquad (5.3.28)$$

with

$$\underline{P}(t) = \text{col.}[P_{h_1}(t), \ldots, P_{h_8}(t), P_{s_g}(t)]$$

$$\underline{W}_i(t) = \text{col.}[Q_i(t), q_i(t)] \qquad i = 1,2,3,5,8 \qquad (5.3.29)$$

$$\underline{W}_4(t) = \text{col.}[Q_4(t), q_4(t), x_1(t), x_2(t), x_3(t)] \qquad (5.3.30)$$

$$\underline{W}_6(t) = \text{col.}[Q_6(t), q_6(t), x_4(t), x_5(t)] \qquad (5.3.31)$$

$$\underline{W}_7(t) = \text{col.}[Q_7(t), q_7(t), x_6(t)] \qquad (5.3.32)$$

and define vector $\underline{L}(t)$ by

$$\underline{L}(t) = \text{col.}[\underline{L}_p(t), \underline{L}_{W_1}(t), \ldots, \underline{L}_{W_8}(t)] \qquad (5.3.33)$$

where

$$\underline{L}_p(t) = \text{col.}[\{n_1(t) - \lambda(t)(1 - B_{10})\}, \ldots,$$

$$\{n_8(t) - \lambda(t)(1 - B_{80})\}, \{\beta - \lambda(t)(1 - B_{90})\}] \qquad (5.3.34)$$

$$\underline{L}_{W_i}(t) = \text{col.}[\dot{m}_i(t), m_i(t) + n_i(t)A_i(t) + p_i(t,\tau_i)]$$

$$i = 1,2,3,5 \qquad (5.3.35)$$

$$\underline{L}_{W_4}(t) = \text{col.}[\dot{m}_4(t), m_4(t) + n_4(t)A_4(t) + p_4(t,\tau_4), 0,0,0] \qquad (5.3.36)$$

$$\underline{L}_{W_6}(t) = \text{col.}[\dot{m}_6(t), m_6(t) + n_6(t)A_6(t) + p_6(t,\tau_6), 0,0] \qquad (5.3.37)$$

$$\underline{L}_{W_7}(t) = \text{col.}[\dot{m}_7(t), m_7(t) + n_7(t)A_7(t), 0] \qquad (5.3.38)$$

$$\underline{L}_{W_8}(t) = \text{col.}[\dot{m}_8(t), m_8(t) + n_8(t)A_8(t)] \qquad (5.3.39)$$

Let the square symmetric matrix $\underline{B}(t)$ be given by:

$$\underline{B}(t) = \text{diag}[\underline{B}_p(t), \underline{B}_{W_1}(t), \ldots, \underline{B}_{W_8}(t)] \qquad (5.3.40)$$

with $$\underline{B}_p(t) = (b_{ij}(t))_{9\times 9} \qquad (5.3.41)$$

$$\begin{aligned} b_{ij}(t) &= \lambda(t)B_{ij} \qquad j,i = 1,9 \quad i = j \neq 9 \\ &= \gamma + \lambda(t)B_{ij} \qquad i = j = 9 \end{aligned} \qquad (5.3.42)$$

$$\underline{B}_{W_i}(t) = \text{diag}[-(\frac{B_i\dot{n}_i(t)}{2} + \theta_i(t,\tau_i)), C_i n_i(t)]$$

$$i = 1,2,3,5 \qquad (5.3.43)$$

$$\underline{B}_{W_4}(t) = \text{diag}[-(\frac{B_4\dot{n}_4(t)}{2} + \theta_4(t,\tau_4)), \underline{B}_{W_{q,x_4}}(t)] \qquad (5.3.44)$$

$$\underline{B}_{W_{c,x_4}}(t) = \begin{bmatrix} C_4 n_4(t) & -\frac{B_4 n_4(t)}{2} & -\frac{B_4 n_4(t)}{2} & -\frac{B_4 n_4(t)}{2} \\ -\frac{B_4 n_4(t)}{2} & r_1(t) & 0 & 0 \\ -\frac{B_4 \dot{n}_4(t)}{2} & 0 & r_2(t) & 0 \\ -\frac{B_4 \dot{n}_4(t)}{2} & 0 & 0 & r_3(t) \end{bmatrix} \qquad (5.3.45)$$

$$\underline{B}_{W_6}(t) = \text{diag}[-(\frac{B_6\hat{n}_6(t)}{2} + \theta_6(t,\tau_6)), \underline{B}_{W_{q,x_6}}(t)] \tag{5.3.46}$$

$$\underline{B}_{W_{q,x_6}}(t) = \begin{bmatrix} C_6 n_6(t) & -\frac{B_6 n_6(t)}{2} & -\frac{B_6 n_6(t)}{2} \\ -\frac{B_6 n_6(t)}{2} & r_4(t) & 0 \\ -\frac{B_6 n_6(t)}{2} & 0 & r_5(t) \end{bmatrix} \tag{5.3.47}$$

$$\underline{B}_{W_7}(t) = \text{diag}[-\frac{B_7\hat{n}_7(t)}{2}, \underline{B}_{W_{q,x_7}}(t)] \tag{5.3.48}$$

$$\underline{B}_{W_{q,x_7}}(t) = \begin{bmatrix} C_7 n_7(t) & -\frac{B_7 n_7(t)}{2} \\ -\frac{B_7 n_7}{2} & r_6(t) \end{bmatrix} \tag{5.3.49}$$

$$\underline{B}_{W_8}(t) = \text{diag}[-\frac{B_8\hat{n}_8(t)}{2}, C_8 n_8(t)] \tag{5.3.50}$$

Using these definitions, the cost functional given by (5.3.27) reduces to:

$$J(\underline{u}) = \int_0^{T_f} \{\underline{L}^T\underline{u}(t) + \underline{u}^T(t)\underline{B}(t)\underline{u}(t)\}dt \tag{5.3.51}$$

Let

$$V^T(t) = \underline{L}^T(t)\underline{B}^{-1}(t) \tag{5.3.52}$$

then J in (5.3.51) reduces to

$$J(\underline{u}) = \int_0^{T_f} [\{\underline{u}(t) + \frac{\underline{V(t)}}{2}\}^T\underline{B}(t)\{\underline{u}(t) + \frac{\underline{V(t)}}{2}\} - \{\frac{\underline{V^T(t)}}{2}\,\underline{B}(t)\,\frac{\underline{V(t)}}{2}\}]dt \tag{5.3.53}$$

The last term in the integrand of (5.3.53) does not depend explicitly on $\underline{u}(t)$, so that it is only necessary to consider

$$J(\underline{u}) = \int_0^{T_f} [\{\underline{u}(t) + \frac{\underline{V}(t)}{2}\}^T \underline{B}(t)\{\underline{u}(t) + \frac{\underline{V}(t)}{2}\}]dt \tag{5.3.54}$$

From (5.3.40) we have

$$\underline{B}^{-1}(t) = \text{diag}[\underline{B}_p{}^{-1}(t), \underline{B}_{W_1}{}^{-1}(t), \ldots, \underline{B}_{W_8}{}^{-1}(t)] \tag{5.3.55}$$

Thus (5.3.52) is rewritten component-wise as:

$$\underline{V}(t) = \text{col.}[\underline{V}_p(t), \underline{V}_{W_1}(t), \ldots, \underline{V}_{W_8}(t)] \tag{5.3.56}$$

with

$$\underline{V}_p(t) = \text{col.}[V_{p_{h_1}}(t), \ldots, V_{p_{h_8}}(t), V_{p_{s_g}}(t)] \tag{5.3.57}$$

$$\underline{V}_{W_i}(t) = \text{col.}[V_{W_{i_1}}(t), V_{W_{i_2}}(t)] \qquad i = 1,2,3,5,8 \tag{5.3.58}$$

$$\underline{V}_{W_4}(t) = \text{col.}[V_{W_{41}}(t), V_{W_{42}}(t), V_{W_{43}}(t), V_{W_{44}}(t), V_{W_{45}}(t)] \tag{5.3.59}$$

$$\underline{V}_{W_6}(t) = \text{col.}[\underline{V}_{W_{61}}(t), \underline{V}_{W_{62}}(t), \underline{V}_{W_{63}}(t), \underline{V}_{W_{64}}(t)] \tag{5.3.60}$$

$$\underline{V}_{W_7}(t) = \text{col.}[V_{W_{71}}(t), V_{W_{72}}(t), \underline{V}_{W_{73}}(t)] \tag{5.3.61}$$

then

$$\underline{V}_p^T(t) = L_p^T \, \underline{B}_p{}^{-1}(t) \tag{5.3.62}$$

$$\underline{V}_{W_i}^T(t) = L_{W_i}^T(t) \, \underline{B}_{W_i}{}^{-1}(t) \tag{5.3.63}$$

Let

$$\underline{B}_p{}^{-1}(t) = (C_{ij}(t))_{9\times9} \tag{5.3.64}$$

and we have

$$\underline{B}_{W_i}{}^{-1}(t) = \text{diag}[-\frac{1}{\frac{B_i\dot{n}_i(t)}{2} + \theta_i(t,\tau_i)}, \frac{1}{C_i n_i(t)}] \qquad i = 1,2,3,5 \tag{5.3.65}$$

$$\underline{B}_{W_4}^{-1}(t) = \text{diag}[-\frac{1}{\frac{B_4\hat{n}_4(t)}{2} + \theta_4(t,\tau_4)}, \underline{B}_{W_{qx_4}}^{-1}(t)] \tag{5.3.66}$$

It can be shown that

$$\underline{B}_{W_{qx_4}}^{-1}(t) = (b_{W_{4_{ij}}}(t)) \tag{5.3.67}$$

is a symmetric matrix whose relevant elements are given by:

$$b_{W_{4_{11}}}(t) = [C_4 n_4(t) - \frac{B_4^2 n_4^2(t)}{4} \sum_{i=1}^{3} \frac{1}{r_i(t)}]^{-1} \tag{5.3.68}$$

$$b_{W_{4_{12}}}(t) = \frac{B_4 n_4(t)}{2r_1(t)} b_{W_{4_{11}}}(t) \tag{5.3.69}$$

$$b_{W_{4_{13}}}(t) = \frac{B_4 n_4(t)}{2r_2(t)} b_{W_{4_{11}}}(t) \tag{5.3.70}$$

$$b_{W_{4_{14}}}(t) = \frac{B_4 n_4(t)}{2r_3(t)} b_{W_{4_{11}}}(t) \tag{5.3.71}$$

Next from (5.3.46)

$$\underline{B}_{W_6}^{-1}(t) = \text{diag}[-\frac{1}{\frac{B_6\hat{n}_6(t)}{2} + \theta_6(t,\tau_6)}, \underline{B}_{W_{qx6}}^{-1}(t)] \tag{5.3.72}$$

with

$$\underline{B}_{W_{qx_6}}^{-1}(t) = (b_{W_{6_{ij}}}(t)) \tag{5.3.73}$$

being a symmetric matrix whose relevant elements are given by:

$$b_{W_{6_{11}}}(t) = [C_6 n_6(t) - \frac{B_6^2 n_6^2(t)}{4} \sum_{i=4}^{5} \frac{1}{r_i(t)}]^{-1} \tag{5.3.74}$$

$$b_{W_{6_{12}}}(t) = \frac{B_6 n_6(t)}{2r_4(t)} b_{W_{6_{11}}}(t) \tag{5.3.75}$$

$$b_{W_{6_{13}}}(t) = \frac{B_6 n_6(t)}{2r_5(t)} b_{W_{6_{11}}}(t) \tag{5.3.76}$$

And from (5.3.48)

$$B_{W_7}^{-1}(t) = \mathrm{diag}[-\frac{2}{B_7 \hat{n}_7(t)}, \underline{B}_{W_{qx_7}}^{-1}(t)] \tag{5.3.77}$$

with

$$B_{W_{qx_7}}^{-1}(t) = (b_{W_{7_{ij}}}(t)) \tag{5.3.78}$$

where

$$b_{W_{7_{11}}}(t) = \frac{r_6(t)}{C_7 n_7(t) r_6(t) - \frac{B_7^2 n_7^2(t)}{4}} \tag{5.3.79}$$

$$b_{W_{7_{12}}}(t) = \frac{B_7 n_7/2}{C_7 n_7(t) r_6(t) - \frac{B_7^2 n_7^2(t)}{4}} \tag{5.3.80}$$

Also from (5.3.50)

$$\underline{B}_{W_8}^{-1}(t) = \mathrm{diag}[-\frac{2}{B_8 \hat{n}_8(t)}, \frac{1}{C_8 n_8(t)}] \tag{5.3.81}$$

Now the components of the vector $\underline{V}(t)$ are obtained as follows: Using (5.3.34) and (5.3.64) in (5.3.62) one obtains:

$$V_{p_{h_j}}(t) = \sum_{i=1}^{8} [n_i(t) - \lambda(t)[1 - B_{io}]]C_{ij}(t)$$
$$+ [\beta - \lambda(t)(1 - B_{go})]C_{gj}(t) \quad j = 1,\ldots,8$$

(5.3.82)

$$V_{P_{sg}}(t) = \sum_{i=1}^{8}[n_i(t) - \lambda(t)(1 - B_{io})]C_{ig}(t)$$

$$+ [\beta - \lambda(t)(1 - B_{go})]C_{gg}(t) \qquad (5.3.83)$$

Substituting (5.3.35) and (5.3.65) in (5.3.62) we have:

$$\underline{V}_{W_i}(t) = \text{col.}[- \frac{\dot{m}_i(t)}{\frac{B_i \dot{n}_i(t)}{2} + \theta_i(t,\tau_i)} ,$$

$$\frac{m_i(t) + n_i(t)A_i(t) + p_i(t,\tau_i)}{C_i n_i(t)}$$

$$i = 1,2,3,5 \qquad (5.3.84)$$

Using (5.3.36), (5.3.66) and (5.3.67) in (5.3.62) one obtain

$$V_{W_{41}}(t) = - \frac{\dot{m}_4(t)}{\frac{B_4 \dot{n}_4(t)}{2} + \theta_4(t,\tau_4)} \qquad (5.3.85)$$

$$V_{W_{42}}(t) = [m_4(t) + n_4(t)A_4(t) + p_4(t,\tau_4)]b_{W_{4_{11}}}(t) \qquad (5.3.86)$$

$$V_{W_{43}}(t) = V_{W_{42}}(t) \frac{b_{W_{4_{12}}}(t)}{b_{W_{4_{11}}}(t)} \qquad (5.3.87)$$

$$V_{W_{44}}(t) = V_{W_{42}}(t) \frac{b_{W_{4_{13}}}(t)}{b_{W_{4_{11}}}(t)} \qquad (5.3.88)$$

$$V_{W_{45}}(t) = V_{W_{42}}(t) \frac{b_{W_{4_{14}}}(t)}{b_{W_{4_{11}}}(t)} \qquad (5.3.89)$$

And from (5.3.37) and (5.3.78) in (5.3.62):

$$V_{W_{61}}(t) = -\frac{\dot{m}_6(t)}{\frac{B_6\dot{n}_6(t)}{2} + \theta_6(t,\tau_6)} \tag{5.3.90}$$

$$V_{W_{62}}(t) = [m_6(t) + n_6(t)A_6(t) + p_6(t,\tau_6)]b_{W_{6_{11}}}(t) \tag{5.3.91}$$

$$V_{W_{63}}(t) = V_{W_{62}}(t)\,\frac{b_{W_{6_{12}}}(t)}{b_{W_{6_{11}}}(t)} \tag{5.3.92}$$

$$V_{W_{64}}(t) = V_{W_{62}}(t)\,\frac{b_{W_{6_{13}}}(t)}{b_{W_{6_{11}}}(t)} \tag{5.3.93}$$

Also substituting (5.3.38) and (5.3.77) into (5.3.62) one obtains:

$$V_{W_{71}}(t) = -\frac{\dot{m}_7(t)}{\frac{B_7\dot{n}_7(t)}{2}} \tag{5.3.94}$$

$$V_{W_{72}}(t) = \frac{[m_7(t) + n_7(t)A_7(t)]r_6(t)}{C_7n_7(t)r_6(t) - \frac{B_7{}^2n_7{}^2}{4}} \tag{5.3.95}$$

$$V_{W_{73}}(t) = V_{W_{72}}\cdot\frac{B_7n_7(t)}{2r_6(t)} \tag{5.3.96}$$

Finally from (5.3.39) and (5.3.81) in (5.3.62) the following is obtained:

$$V_{W_8}(t) = \text{col.}\left[-\frac{2\dot{m}_8(t)}{B_8\dot{n}_8(t)},\ \frac{m_8(t) + n_8(t)A_8(t)}{C_8n_8(t)}\right] \tag{5.3.97}$$

Thus the components of the vector $\underline{V}(t)$ are determined by the above equations.

The problem formulated thus far is that of minimizing (5.3.54) subject to (5.3.3). Define the 8x1 column vector:

$$\underline{b} = \text{col.}[b_1,\ldots,b_8] \tag{5.3.98}$$

and the (8x21) matrix $\underline{K}^T$ as:

$$\underline{K}^T = \begin{bmatrix} \underline{0} & \underline{K}_1^T & \underline{0} & & \\ \underline{0} & \underline{0} & \underline{K}_2^T & & \\ & & & & \\ & & & & \underline{K}_8^T \end{bmatrix} \tag{5.3.99}$$

with

$$\underline{K}_i^T = [0,1] \qquad i = 1,2,3,5,8 \tag{5.3.100}$$

$$\underline{K}_4^T = [0,1,0,0,0] \tag{5.3.101}$$

$$\underline{K}_6^T = [0,1,0,0] \tag{5.3.102}$$

$$\underline{K}_7^T = [0,1,0] \tag{5.3.103}$$

Then (5.3.3) reduces to

$$\underline{b} = \int_0^{T_f} \underline{K}^T \underline{u}(s)ds \tag{5.3.104}$$

The control vector $\underline{u}(t)$ is considered an element of the Hilbert space $L_{2,\underline{B}}^{(21)}[0,T_f]$ of the 21-vector valued square integrable functions defined on $[0,T_f]$ endowed with the inner product definition:

$$<\underline{V}(t),\underline{u}(t)> = \int_0^{T_f} \underline{V}^T(t)\underline{B}(t)\underline{u}(t)dt \tag{5.3.105}$$

for every $\underline{V}(t)$ and $\underline{u}(t)$ in $L_{2,\underline{B}}^{(21)}[0,T_f]$, provided that $\underline{B}(t)$ is positive definite.

The given vector $\underline{b}$ is considered an element of the Real space R^8 with the Euclidean inner product definition:

$$<\underline{X},\underline{Y}> = \underline{X}^T\underline{Y} \tag{5.3.106}$$

for every $\underline{X}$ and $\underline{Y}$ in R.

Equation (5.3.104) defines a bounded linear transformation $t:\ L^{(21)}_{2,\underline{B}}[0,T_f]\to R^8$. This can be expressed as:

$$\underline{b} = T[\underline{u}(t)] \tag{5.3.107}$$

and the cost functional given by (5.3.54) reduces to:

$$J[\underline{u}(t)] = ||\underline{u}(t) + \frac{\underline{V}(t)}{2}||^2 \tag{5.3.108}$$

Finally it is necessary only to minimize:

$$J[\underline{u}(t)] = ||\underline{u}(t) + \frac{\underline{V}(t)}{2}|| \tag{5.3.109}$$

subject to

$$\underline{b} = T[\underline{u}(t)] \quad \text{for a given } \underline{b} \text{ in R.}$$

5.3.3 The Optimal Solution:

The optimal solution to the problem formulated in the previous subsection, using the results of Chapter 2 is:

$$\underline{u}_\xi(t) = T^\dagger[\underline{b} + T(\frac{\underline{V}(t)}{2})] - \frac{\underline{V}(t)}{2} \tag{5.3.110}$$

where $T^\dagger$ is obtained as follows:

T^*, the adjoint of T, is obtained using the identity:

$$\underline{<\underline{\xi},Tu>}_{R^8} = \underline{<T^*\xi,\underline{u}>}_{L^{21}_{2,B}[0,T_f]} \tag{5.3.111}$$

Let

$$\underline{\xi} = \text{col.}[\xi_1,\ldots,\xi_8] \tag{5.3.112}$$

$$\underline{T^*\xi} = \text{col.}[\underline{T}_p,\underline{T}_{W_1},\ldots,\underline{T}_{W_8}] \tag{5.3.113}$$

$$\underline{T}_p = \text{col.}[t_{p_1},\ldots,t_{p_g}] \tag{5.3.114}$$

$$\underline{T}_{W_i} = \text{col.}[t_{W_{i_1}},t_{W_{i_2}}] \qquad i = 1,2,3,5,8 \tag{5.3.115}$$

$$\underline{T}_{W_4} = \text{col.}[t_{W_{41}},t_{W_{42}},t_{W_{43}},t_{W_{44}},t_{W_{45}}] \tag{5.3.116}$$

$$\underline{T}_{W_6} = \text{col.}[t_{W_{61}}, t_{W_{62}}, t_{W_{63}}, t_{W_{64}}] \tag{5.3.117}$$

$$\underline{T}_{W_7} = \text{col.}[t_{W_{71}}, t_{W_{72}}, t_{W_{73}}] \tag{5.3.118}$$

$$\underline{\phi}_i = \text{col.}[0, \xi_i] \qquad i = 1,2,3,5,8 \tag{5.3.119}$$

$$\underline{\phi}_4 = \text{col.}[0, \xi_4, 0, 0, 0] \tag{5.3.120}$$

$$\underline{\phi}_6 = \text{col.}[0, \xi_6, 0, 0] \tag{5.3.121}$$

$$\underline{\phi}_7 = \text{col.}[0, \xi_7, 0, 0] \tag{5.3.122}$$

Then in R^8 one has

$$\langle \underline{\xi}, \underline{Tu} \rangle_{R^8} = \int_0^{T_f} \sum_{i=1}^{8} \underline{\phi}_i^T \, \underline{W}_i \, dt \tag{5.3.123}$$

and in $L^{21}_{2,B}[0,T_f]$

$$\langle \underline{T}^* \underline{\xi}, \underline{u} \rangle = \int_0^{T_f} \{\underline{T}_p \underline{B}_p(t) \underline{P}(t) + \sum_{i=1}^{8} \underline{T}_{W_i} \underline{B}_{W_i}(t) \underline{W}_i(t)\} dt \tag{5.3.124}$$

Thus the equality (5.3.111) reduces to:

$$\int_0^{T_f} [\underline{T}_p \underline{B}_p(t) \underline{P}(t) + \sum_{i=1}^{8} \underline{T}_{W_i} \underline{B}_{W_i}(t) \underline{W}_i(t)] dt$$

$$= \int_0^{T_f} \sum_{i=1}^{8} \underline{\phi}_i^T \, \underline{W}_i \, dt \tag{5.3.125}$$

This is satisfied for the choice:

$$\underline{T}_p = \underline{0} \tag{5.3.126}$$

$$\underline{T}_{W_i} = \underline{\phi}_i^T \, \underline{B}_{W_i}^{-1}(t) \qquad i = 1,\ldots,8 \tag{5.3.127}$$

Substituting the inverse matrices in (5.3.127) this reduces to:

$$\underline{T}_{W_i} = \text{col.}[0, \frac{\xi_i}{C_i n_i(t)}] \qquad i = 1,2,3,5,8 \tag{5.3.128}$$

$$\underline{T}_{W_4} = \text{col.}[0, \xi_4 b_{W_{4_{11}}}(t), \xi_4 b_{W_{4_{12}}}(t), \xi_4 b_{W_{4_{13}}}(t),$$

$$\xi_4 b_{W_{4_{14}}}(t)] \tag{5.3.129}$$

$$\underline{T}_{W_6} = \text{col.}[0, \xi_6 b_{W_{6_{11}}}(t), \xi_6 b_{W_{6_{12}}}(t), \xi_6 b_{W_{6_{13}}}(t)] \qquad (5.3.130)$$

$$\underline{T}_{W_7} = \text{col.}[0, \xi_7 b_{W_{7_{11}}}(t), \xi_7 b_{W_{7_{12}}}(t)] \qquad (5.3.131)$$

This completely specifies $\underline{T^*\xi}$ given by (5.3.113).

The operator J is evaluated from:

$$J[\underline{\xi}] = T[\underline{T^*\xi}] \qquad (5.3.132)$$

Using (5.3.104) and (5.3.113) this reduces to:

$$\begin{aligned} J(\underline{\xi}) = \text{col.}[&\{\xi_1 \int_0^{T_f} \frac{1}{C_1 n_1(t)}\, dt\}, \{\xi_2 \int_0^{T_f} \frac{1}{C_2 n_2(t)}\, dt\}, \\ &\{\xi_3 \int_0^{T_f} \frac{1}{C_3 n_3(t)}\, dt\}, \{\xi_4 \int_0^{T_f} b_{W_{4_{11}}}(t) dt\}, \\ &\{\xi_5 \int_0^{T_f} \frac{1}{C_5 n_5(t)}\, dt\}, \{\xi_6 \int_0^{T_f} b_{W_{6_{11}}}(t) dt\}, \\ &\{\xi_7 \int_0^{T_f} b_{W_{7_{11}}}(t) dt\}, \{\xi_8 \int_0^{T_f} \frac{1}{C_8 n_8(t)}\, dt\} \end{aligned} \qquad (5.3.133)$$

or

$$J[\underline{\xi}] = \underline{\Lambda}\ \underline{\xi} \qquad (5.3.134)$$

with

$$\begin{aligned} \underline{\Lambda} = \text{diag}[&\int_0^{T_f} \frac{1}{C_1 n_1(t)}\, dt, \int_0^{T_f} \frac{1}{C_2 n_2(t)}\, dt, \int_0^{T_f} \frac{1}{C_3 n_3(t)}\, dt, \\ &\int_0^{T_f} b_{W_{4_{11}}}(t), \int_0^{T_f} \frac{1}{C_5 n_5(t)}\, dt, \int_0^{T_f} b_{W_{6_{11}}}(t) dt, \\ &\int_0^{T_f} b_{W_{7_{11}}}(t) dt, \int_0^{T_f} \frac{1}{C_8 n_8(t)}\, dt] \end{aligned} \qquad (5.3.135)$$

Thus the operation $\underline{J^{-1}\xi}$ is given by

$$\underline{J^{-1}\xi} = \underline{\Lambda}^{-1}\ \underline{\xi} \tag{5.3.136}$$

This yields the pseudo-inverse operation given by

$$T^{\dagger}\xi = T^{*}[\underline{J^{-1}\xi}] \tag{5.3.137}$$

as:

$$\underline{T^{\dagger}\xi}|_{\underline{P}} = \underline{0} \tag{5.3.138}$$

$$\underline{T^{\dagger}\xi}|_{W_i} = \text{col.}[0, \frac{\xi_i}{n_i(t)\int_0^{T_f} \frac{1}{n_i(t)}\, dt}]$$

$$i = 1,2,3,5,8 \tag{5.3.139}$$

$$\underline{T^{\dagger}\xi}|_{W_4} = \text{col.}[0, \frac{b_{W_{4_{11}}}\ \xi_4}{\int_0^{T_f} b_{W_{4_{11}}}(t)dt}, \frac{b_{W_{4_{12}}}\ \xi_4}{\int_0^{T_f} b_{W_{4_{11}}}(t)dt},$$

$$\frac{b_{W_{4_{13}}}(t)\xi_4}{\int_0^{T_f} b_{W_{4_{11}}}(t)dt}, \frac{b_{W_{4_{14}}}(t)\xi_4}{\int_0^{T_f} b_{W_{4_{11}}}(t)dt}] \tag{5.3.140}$$

$$\underline{T^{\dagger}\xi}|_{W_6} = \text{col.}[0, \frac{b_{W_{6_{11}}}(t)\xi_6}{\int_0^{T_f} b_{W_{6_{11}}}(t)dt}, \frac{b_{W_{6_{12}}}(t)\xi_6}{\int_0^{T_f} b_{W_{6_{11}}}(t)dt},$$

$$\frac{b_{W_{6_{13}}}(t)\xi_6}{\int_0^{T_f} b_{W_{6_{11}}}(t)dt}] \tag{5.3.141}$$

$$\underline{T^{\dagger}\xi}\Big|_{W_7} = \text{col.}\left[0, \frac{b_{W_{7_{11}}}(t)\xi_7}{\int_0^{T_f} b_{W_{7_{11}}}(t)dt}, \frac{b_{W_{7_{12}}}(t)\xi_7}{\int_0^{T_f} b_{W_{7_{11}}}(t)dt}\right] \tag{5.3.142}$$

In the expression for the optimal solution (5.3.110) let:

$$\underline{n} = \underline{b} + T\left(\frac{V(t)}{2}\right) \tag{5.3.143}$$

This is an (8x1) vector whose components are found to be:

$$n_i = b_i + \int_0^{T_f} \frac{m_i(t) + n_i(t)A_i(t) + p_i(t,\tau_i)}{2C_i n_i(t)} dt$$

$$i = 1,2,3,5 \tag{5.3.144}$$

$$n_4 = b_4 + \frac{1}{2}\int_0^{T_f} [m_4(t) + n_4(t)A_4(t) + P_4(t,\tau_4)]$$

$$b_{W_{4_{11}}}(t)dt \tag{5.3.145}$$

$$n_6 = b_6 + \frac{1}{2}\int_0^{T_f} [m_6(t) + n_6(t)A_6(t) + p_6(t,\tau_6)]$$

$$b_{W_{6_{11}}}(t)dt \tag{5.3.146}$$

$$n_7 = b_7 + \frac{1}{2}\int_0^{T_f} \frac{[m_7(t) + n_7(t)A_7(t)]}{C_7 n_7(t) r_6(t) - \frac{B_7^{\,2} n_7^{\,2}(t)}{4}} r_6(t)dt \tag{5.3.147}$$

$$n_8 = b_8 + \frac{1}{2}\int_0^{T_f} \frac{[m_8(t) + n_8(t)A_8(t)]}{C_8 n_8(t)} dt \tag{5.3.148}$$

Thus replacing $\underline{\xi}$ components in (5.3.138) through (5.3.143) by components of $\underline{n}$ as given in (5.3.144) through (5.3.148) one obtains

$$\underline{T}^{\dagger}\underline{\xi}\big|_{\underline{P}} = \underline{0} \tag{5.3.149}$$

$$\underline{T}^{\dagger}\underline{\eta}\big|_{\underline{W}_i} = \text{col.}[0, \frac{\eta_i}{n_i(t)\int_0^{T_f} \frac{1}{n_i(t)}\,dt}] \quad i = 1,2,3,5,8 \tag{5.3.150}$$

$$\underline{T}^{\dagger}\underline{\eta}\big|_{\underline{W}_4} = \text{col.}[0, \frac{b_{W_{4_{11}}}(t)\eta_4}{\int_0^{T_f} b_{W_{4_{11}}}(t)dt}, \frac{b_{W_{4_{12}}}(t)\eta_4}{\int_0^{T_f} b_{W_{4_{11}}}(t)dt},$$

$$\frac{b_{W_{4_{13}}}(t)\eta_4}{\int_0^{T_f} b_{W_{4_{11}}}(t)dt}, \frac{b_{W_{4_{14}}}(t)\eta_4}{\int_0^{T_f} b_{W_{4_{11}}}(t)dt}] \tag{5.3.151}$$

$$\underline{T}^{\dagger}\underline{\eta}\big|_{\underline{W}_6} = \text{col.}[0, \frac{b_{W_{6_{11}}}(t)\eta_6}{\int_0^{T_f} b_{W_{6_{11}}}(t)dt}, \frac{b_{W_{6_{12}}}(t)\eta_6}{\int_0^{T_f} b_{W_{6_{11}}}(t)dt},$$

$$\frac{b_{W_{6_{13}}}(t)\eta_6}{\int_0^{T_f} b_{W_{6_{11}}}(t)dt}] \tag{5.3.152}$$

$$\underline{T}^{\dagger}\underline{\eta}\big|_{\underline{W}_7} = \text{col.}[0, \frac{b_{W_{7_{11}}}(t)\eta_7}{\int_0^{T_f} b_{W_{7_{11}}}(t)dt}, \frac{b_{W_{7_{12}}}\eta_7}{\int_0^{T_f} b_{W_{7_{11}}}(t)dt}] \tag{5.3.153}$$

Now the optimal solution given by (5.3.110) is obtained component-wise as:

$$\underline{P}_{\xi}(t) = -\frac{\underline{V}_p(t)}{2} \qquad (5.3.154)$$

$$Q_{\xi_i}(t) = \frac{\dot{m}_i(t)}{B_i\dot{n}_i(t) + 2\theta_i(t,\tau_i)} \qquad i = 1,\ldots,6 \qquad (5.3.155)$$

$$Q_{\xi_i}(t) = \frac{\dot{m}_i(t)}{B_i\dot{n}_i(t)} \qquad i = 7,\ 8 \qquad (5.3.156)$$

$$q_{\xi_i}(t) = -\frac{[m_i(t) + n_i(t)A_i(t) + p_i(t,\tau_i)]}{2C_i n_i(t)} + \frac{b_i + \int_0^{T_f} \frac{m_i(t) + n_i(t)A_i(t) + p_i(t,\tau_i)}{2C_i n_i(t)}\,dt}{n_i(t)\int_0^{T_f} \frac{1}{n_i(t)}\,dt}$$

$$i = 1,2,3,5 \qquad (5.3.157)$$

$$q_{\xi_4}(t) = -\frac{b_{W_{4_{11}}}(t)}{2}[m_4(t) + n_4(t)A_4(t) + p_4(t,\tau_4)] + \frac{b_{W_{4_{11}}}(t)[b_4 + \frac{1}{2}\int_0^{T_f} b_{W_{4_{11}}}(t)[m_4(t) + n_4(t)A_4(t) + p_4(t,\tau_4)]dt}{\int_0^{T_f} b_{W_{4_{11}}}(t)dt}$$

$$(5.3.158)$$

$$q_{\xi_6}(t) = -\frac{b_{W_{6_{11}}}(t)}{2}[m_6(t) + n_6(t)A_6(t) + p_6(t,\tau_6)]$$

$$+ \frac{b_{W_{6_{11}}}(t)[b_6 + \frac{1}{2}\int_o^{T_f} b_{W_{6_{11}}}(t)[m_6(t) + n_6(t)A_6(t) + p_6(t,\tau_6)]dt}{\int_o^{T_f} b_{W_{6_{11}}}(t)dt} \tag{5.3.159}$$

$$q_{\xi_7}(t) = -\frac{b_{W_{7_{11}}}(t)[m_7(t) + n_7(t)A_7(t)]}{2}$$

$$+ \frac{b_{W_{7_{11}}}(t)[b_7 + \frac{1}{2}\int_o^{T_f}[m_7(t) + n_7(t)A_7(t)]b_{W_{7_{11}}}(t)dt]}{\int_o^{T_f} b_{W_{7_{11}}}(t)dt} \tag{5.3.160}$$

$$q_{\xi_8}(t) = -\frac{[m_8(t) + n_8(t)A_8(t)]}{2C_8n_8(t)}$$

$$+ \frac{[b_8 + \frac{1}{2}\int_o^{T_f}\frac{m_8(t) + n_8(t)A_8(t)}{C_8n_8(t)}dt]}{n_8(t)\int_o^{T_f}\frac{1}{n_8(t)}dt} \tag{5.3.161}$$

$$x_{\xi_1}(t) = \frac{B_4n_4(t)}{2r_1(t)}q_{\xi_4}(t) \tag{5.3.162}$$

$$x_{\xi_2}(t) = \frac{B_4n_4(t)}{2r_2(t)}q_{\xi_4}(t) \tag{5.3.163}$$

$$x_{\xi_3}(t) = \frac{B_4n_4(t)}{2r_3(t)}q_{\xi_4}(t) \tag{5.3.164}$$

$$x_{\xi_4}(t) = \frac{B_6n_6(t)}{2r_4(t)}q_{\xi_6}(t) \tag{5.3.165}$$

$$x_{\xi_5}(t) = \frac{B_6 n_6(t)}{2r_5(t)} q_{\xi_6}(t) \tag{5.3.166}$$

$$x_{\xi_6}(t) = \frac{B_7 n_7(t)}{2r_6(t)} q_{\xi_7}(t) \tag{5.3.167}$$

5.3.4 The Modified Optimal Solution

Here, the pseudo-control variables x(t) and q(t) will be eliminated together with the associated multipliers. Rewrite (5.3.157) as:

$$2C_i n_i(t)\dot{Q}_{\xi_i}(t) + m_i(t) + n_i(t)A_i(t) + p_i(t,\tau_i) = e_i$$

$$i = 1,2,3,5 \qquad (5.3.168)$$

(5.3.158) as:

$$\frac{2}{b_{W_{4_{11}}}(t)}\dot{Q}_{\xi_4}(t) + m_4(t) + n_4(t)A_4(t) + p_4(t,\tau_4) = e_4 \tag{5.3.169}$$

(5.3.159) as:

$$\frac{2}{b_{W_{6_{11}}}(t)}\dot{Q}_{\xi_6}(t) + m_6(t) + n_6(t)A_6(t) + p_6(t,\tau_6) = e_6 \tag{5.3.170}$$

(5.3.160) as:

$$\frac{2}{b_{W_{7_{11}}}(t)}\dot{Q}_{\xi_7}(t) + m_7(t) + n_7(t)A_7(t) = e_7 \tag{5.3.171}$$

and (5.3.161) as:

$$2C_8 n_8(t)\dot{Q}_{\xi_8}(t) + m_8(t) + n_8(t)A_8(t) = e_8 \tag{5.3.172}$$

where e_i are the constants in the corresponding $q_{i_\xi}(t)$ expressions given in (5.3.157) through (5.3.161). Differentiating (5.3.168) through (5.3.172) one obtains:

$$\frac{d}{dt}\left[2C_i n_i(t)\dot{Q}_{\xi_i}(t) + n_i(t)A_i(t)\right] + \dot{m}_i(t) + \dot{p}_i(t,\tau_i) = 0$$

$$i = 1,2,3,5 \qquad (5.3.173)$$

$$\frac{d}{dt}\left[\frac{2}{b_{W_{4_{11}}}(t)}\dot{Q}_{\xi_4}(t) + n_4(t)A_4(t)\right] + \dot{m}_4(t) + \dot{p}_4(t,\tau_4) = 0 \qquad (5.3.174)$$

$$\frac{d}{dt}\left[\frac{2}{b_{W_{6_{11}}}(t)}\dot{Q}_{\xi_6}(t) + n_6(t)A_6(t)\right] + \dot{m}_6(t) + \dot{p}_6(t,\tau_6) = 0 \qquad (5.3.175)$$

$$\frac{d}{dt}\left[\frac{2}{b_{W_{7_{11}}}(t)}\dot{Q}_{\xi_7}(t) + n_7(t)A_7(t)\right] + \dot{m}_7(t) = 0 \qquad (5.3.176)$$

$$\frac{d}{dt}\left[2C_8 n_8(t)\dot{Q}_{\xi_8}(t) + n_8(t)A_8(t)\right] + \dot{m}_8(t) = 0 \qquad (5.3.177)$$

Substituting for $\dot{p}_i(t,\tau_i)$ from (5.3.25) and (5.3.26) in (5.3.173) through (5.3.175) one obtains:

$$\frac{d}{dt}\left[2C_i n_i(t)\dot{Q}_{\xi_i}(t) + n_i(t)A_i(t)\right] + \dot{m}_i(t) + 2\psi(\tau_i,\tau_i)$$

$$\theta_i(t,\tau_i) = 0 \qquad i = 1,2,3,5 \quad (5.3.178)$$

$$\frac{d}{dt}\left[\frac{2}{b_{W_{4_{11}}}(t)}\dot{Q}_{\xi_4}(t) + n_4(t)A_4(t)\right] + \dot{m}_4(t) + 2\psi_4(\tau_4,\tau_4)$$

$$\theta_4(t,\tau_4) = 0 \qquad (5.3.179)$$

$$\frac{d}{dt}\left[\frac{2}{b_{W_{6_{11}}}(t)}\dot{Q}_{\xi_6}(t) + n_6(t)A_6(t)\right] + \dot{m}_6(t) + 2\psi(\tau_6,\tau_6)$$

$$\theta_6(t,\tau_6) = 0 \qquad (5.3.180)$$

Substituting (5.3.155) for $\dot{n}_i(t)$ in (5.3.178) through (5.3.180) the following is obtained:

$$\frac{d}{dt}[2C_i n_i(t)\dot{Q}_{\xi_i}(t) + n_i(t)A_i(t)] + B_i\dot{n}_i(t)Q_{\xi_i}(t)$$

$$+ 2\theta_i(t,\tau_i)[\psi_i(\tau_i,\tau_i) + Q_{\xi_i}(t)] = 0$$

$$i = 1,2,3,5 \qquad (5.3.181)$$

$$\frac{d}{dt}[\frac{2}{b_{W_{4_{11}}}(t)}\dot{Q}_{\xi_4}(t) + n_4(t)A_4(t)] + B_4\dot{n}_4(t)Q_{\xi_4}(t)$$

$$+ 2\theta_4(t,\tau_4)[\psi_4(\tau_4,\tau_4) + Q_{\xi_4}(t)] = 0 \qquad (5.3.182)$$

$$\frac{d}{dt}[\frac{2}{b_{W_{6_{11}}}(t)}\dot{Q}_{\xi_6}(t) + n_6(t)A_6(t)] + B_6\dot{n}_6(t)Q_{\xi_6}(t)$$

$$+ 2\theta_6(t,\tau_6)[\psi_6(\tau_6,\tau_6) + Q_{\xi_6}(t)] = 0 \qquad (5.3.183)$$

Consider the constraints (5.3.11) and (5.3.12)

$$x_i(t) = \psi_i(t,\tau_i) \qquad t \leq \tau_i \qquad (5.3.184)$$

$$x_i(t) = \psi_i(\tau_i,\tau_i) + Q_i(t-\tau_i), \qquad \tau_i < t \leq T_f \qquad (5.3.185)$$

Rewriting (5.3.185) for a time advance τ_j one has:

$$x_i(t+\tau_j) = \psi_i(\tau_j,\tau_j) + Q_i(t) \qquad 0 \leq t \leq T_f-\tau_i \qquad (5.3.186)$$

The equations (5.3.162) through (5.3.167) can be rewritten as:

$$2r_i(t+\tau_i)x_i(t+\tau_i) = B_j n_j(t+\tau_i)\dot{Q}_j(t+\tau_j)$$

$$0 \leq t \leq T_f-\tau_i \qquad (5.3.187)$$

where for i = 1,2,3 we have j = 4, for i = 4,5 we have j = 6 and for i = 6, j = 7.

And by definition (5.3.26)

$$\theta_i(t,\tau_i) = r_i(t+\tau_i) \qquad 0 \le t \le T_f-\tau_i \qquad (5.3.188)$$

$$= 0 \qquad T_f - \tau_i < t \le T_f$$

Thus (5.3.181) through (5.3.183) reduce to:

$$\frac{d}{dt}[2C_i n_i(t)\dot{Q}_{\xi_i}(t) + n_i(t)A_i(t)] + B_i\dot{n}_i(t)Q_{\xi_i}(t)$$

$$+ B_j n_j(t+\tau_i)\dot{Q}_{\xi_j}(t+\tau_i) = 0$$

$$i = 1,2,3 \qquad j = 4$$

$$i = 5 \qquad j = 6 \qquad 0 \le t \le T_f-\tau_i$$

$$(5.3.189)$$

$$\frac{d}{dt}[\frac{2}{b_{W_{4_{11}}}(t)}\dot{Q}_{\xi_4}(t) + n_4(t)A_4(t)] + B_4\dot{n}_4(t)Q_{\xi_4}(t)$$

$$+ B_6 n_6(t+\tau_4)\dot{Q}_{\xi_6}(t+\tau_4) = 0$$

$$0 \le t \le T_f-\tau_4 \qquad (5.3.190)$$

$$\frac{d}{dt}[\frac{2}{b_{W_{6_{11}}}(t)}\dot{Q}_{\xi_6}(t) + n_6(t)A_6(t)] + B_6\dot{n}_6(t)Q_{\xi_6}(t)$$

$$+ B_7 n_7(t+\tau_6)\dot{Q}_{\xi_7}(t+\tau_6) = 0 \qquad (5.3.191)$$

$$0 \le t \le T_f-\tau_6$$

Note that for $t \,\varepsilon(T_f-\tau_j,T_f]$, equations (5.3.189) through (5.3.191) hold true with exception that the last term in the left-hand side (time

lead) disappears. Note that (5.3.189) now depends only on $n_i(t)$ and $Q_i(t)$, so that this equation is the modified optimal equation for this type of plants. However, (5.3.150) and (5.3.191) contain the $r_i(t)$ functions implicitly. This will be considered as follows:

Consider (5.3.68) rewritten as:

$$[b_{W_{4_{11}}}(t)]^{-1} = [C_4 n_4(t) - \frac{B_4^2 n_4^2(t)}{4} \sum_{i=1}^{3} \frac{1}{r_i(t)}] \tag{5.3.192}$$

substituting (5.3.162) through (5.3.164) in (5.3.192) one obtains:

$$[b_{W_{4_{11}}}(t)]^{-1} = [C_4 n_4(t) - \frac{B_4 n_4(t)}{2Q_{\xi_4}(t)} \sum_{i=1}^{3} x_{\xi_i}(t)] \tag{5.3.193}$$

Also (5.3.74) is rewritten as:

$$[b_{W_{6_{11}}}(t)]^{-1} = [C_6 n_6(t) - \frac{B_6 n_6(t)}{2Q_{\xi_6}(t)} \sum_{i=4}^{5} x_{\xi_i}(t)] \tag{5.3.194}$$

Thus (5.3.190) and (5.3.191) are given by:

$$\frac{d}{dt}[2C_4 n_4(t)\dot{Q}_{\xi_4}(t) + n_4(t)A_4(t) - B_4 n_4(t) \sum_{i=1}^{3} x_{\xi_i}(t)]$$

$$+ B_4 \dot{n}_4(t) Q_{\xi_4}(t) + B_6 n_6(t+\tau_4)\dot{Q}_{\xi_6}(t+\tau_4) = 0$$

$$0 \leq t \leq T_f - \tau_4 \tag{5.3.195}$$

$$\frac{d}{dt}[2C_6 n_6(t)\dot{Q}_{\xi_6}(t) + n_6(t)A_6(t) - B_6 n_6(t) \sum_{i=4}^{5} x_{\xi_i}(t)]$$

$$+ B_6 \dot{n}_6(t) Q_{\xi_6}(t) + B_7 n_7(t+\tau_6)\dot{Q}_{\xi_7}(t+\tau_6) = 0$$

$$0 \leq t \leq T_f - \tau_6 \tag{5.3.196}$$

Note that by (5.3.11) and (5.3.12) we have

$$x_i(t) = \psi_i(t,\tau_i) \qquad t \le \tau_i \tag{5.3.197}$$

$$= \psi_i(\tau_i,\tau_i) + Q_i(t-\tau_i) \quad t > \tau_i \tag{5.3.198}$$

so that (5.3.195) and (5.3.196) are in terms of $n_i(t)$ and $Q_i(t)$. Finally substituting (5.3.156) in (5.3.176) and (5.3.177) one obtains

$$\frac{d}{dt}[2C_7 n_7(t)\dot{Q}_{\xi_7}(t) - B_7 n_7(t)x_6(t) + n_7(t)A_7(t)]$$

$$+ B_7\dot{n}_7(t)Q_{7_\xi}(t) = 0 \tag{5.3.199}$$

$$\frac{d}{dt}[2C_8 n_8(t)\dot{Q}_{\xi_8}(t) + n_8(t)A_8(t)] + B_8\dot{n}_8(t)Q_{\xi_8}(t) = 0 \tag{5.3.200}$$

The modified optimal equations (5.3.189), (5.3.195), (5.3.196), (5.3.199) and (5.3.200) can be rewritten in the general form

$$\frac{d}{dt}[2C_i n_i(t)\dot{Q}_{\xi_i}(t) + n_i(t)A_i(t)] + B_i\dot{n}_i(t)Q_{\xi_i}(t) + g_i(t) = 0 \tag{5.3.201}$$

Here the g_i's are given by:

$$g_i(t) = B_4 n_4(t+\tau_i)\dot{Q}_{\xi_4}(t+\tau_i) \qquad 0 \le t \le T_f-\tau_i,\quad i = 1,2,3 \tag{5.3.202}$$

$$g_i(t) = 0 \qquad T_f-\tau_i < t \le T_f,\quad i = 1,2,3 \tag{5.3.203}$$

$$g_5(t) = B_6 n_6(t+\tau_5)\dot{Q}_{\xi_6}(t+\tau_5) \qquad 0 \le t \le T_f-\tau_6 \tag{5.3.204}$$

$$g_5(t) = 0 \qquad T_f-\tau_6 < t \le T_f \tag{5.3.205}$$

$$g_4(t) = B_6 n_6(t+\tau_4)\dot{Q}_{\xi_6}(t+\tau_4) - \frac{d}{dt}[B_4 n_4(t) \sum_{i=1}^{3} x_i(t,\tau_i)]$$

$$0 \leq t \leq T_f - \tau_4 \qquad (5.3.206)$$

$$= - \frac{d}{dt}[B_4 n_4(t) \sum_{i=1}^{3} x_i(t,\tau_1)]$$

$$T_f - \tau_4 < t \leq T_f \qquad (5.3.207)$$

$$g_6(t) = B_7 n_7(t+\tau_6)\dot{Q}_{\xi_7}(t+\tau_6) - \frac{d}{dt}[B_6 n_6(t) \sum_{i=4}^{5} x_{\xi_i}(t,\tau_i)]$$

$$0 \leq t \leq T_f - \tau_6 \qquad (5.3.208)$$

$$g_6(t) = - \frac{d}{dt}[B_6 n_6(t) \sum_{i=4}^{5} x_{\xi_i}(t,\tau_i)]$$

$$T_f - \tau_6 < t \leq T_f \qquad (5.3.209)$$

$$g_7(t) = - \frac{d}{dt}[B_7 n_7(t) x_6(t)] \qquad (5.3.210)$$

$$g_8(t) = 0 \qquad (5.3.211)$$

CHAPTER VI

OPTIMAL HYDRO-THERMAL POWER FLOW

6.1 Background

In this chapter extensions of the problems considered in the previous chapters are made. Here, the active power balance equation used in the previous formulations is replaced by the load flow equations. This is as far as the electric network variables are concerned. One more consideration is the inclusion of a reliability objective in the cost functional. Moreover, realistic inequality constraints imposed on the electric variables are considered. In the final section of this chapter a practical form of the reservoir (trapezoidal) is considered. Furthermore the effect of efficiency variations with the active power generation at the hydro-plants is included. The problem of implementing the optimum generation schedules is illustrated by way of an example.

The extension of the existing economy dispatch solutions to include the exact model of the transmission network is due to Carpentier [58]. The resulting optimization problem was shown to be one of nonlinear programming. Necessary conditions for optimality were derived using the nonlinear programming techniques. In their paper [59], J. Peschon and his associates presented the general problem considered by Carpentier for an all-thermal system. Another important contribution is that of H.W. Dommel and W.F. Tinney [60]. Here optimal power flow solutions are

obtained for an all-thermal system. The method is based on power flow solution by Newton's Method [61], and a gradient adjustment algorithm for obtaining the minimum is employed.

A unified approach to load-flow, minimum loss, and economic dispatching problems was presented by A.M. Sasson [62]. Here investigation of the application of various nonlinear programming methods to the problem was considered. In [63], A.H. El-Abiad and F.J. Jaimes presented a variational method to solve the optimal load flow problem. It is noted that these two works were also concerned with all-thermal systems.

The problem of power systems reliability motivated the work of R.L. Sullivan and O.I. Elgerd [64]. An effort to define a reliability objective in terms of the system's reactive power generations was made. The basic idea of their work was to optimally distribute the reactive power generation between the system generators.

The work by C.M. Shen and M.A. Laughton [65] was of the same nature as those previously mentioned. The main contribution here was exploring the problem of existence and uniqueness of the optimal solution using nonlinear programming techniques. The problem of a hydro-thermal system with negligible head variations was solved in [66]. Here a discrete formulation was adopted and the problem is solved using the nonlinear programming techniques. The discretization process makes the problem a one of a large dimension. However a method of splitting the problem into ones of smaller dimension was proposed.

A dual linear programming formulation was given in [67]. Here a fast solution can be obtained for the problem of an all-thermal system under inequality constraints. This was a contribution to the online

dispatching problem. The need for including objectives other than economy was given in [68]. Here a minimum emission dispatch problem was considered. A variational technique is employed to obtain the solution.

A generalized reduced gradient technique is used for obtaining optimal-power flow solutions in [69]. This represents the best method to date for solving problems of very high dimension.

6.2 Statement of the Problem

A hydro-thermal power system is considered. The system is assumed to have N_g generating plants (generator buses). There are N_h hydro-plants and $(N_g - N_h)$ thermal plants. The system's electric network is represented by N buses (or nodes) and these are connected by branches or lines having conductance G^{ij} and admittance B^{ij}. Connected between bus i and neutral is a branch having conductance G^{io} and admittance B^{io}. This is required for the equivalent π representation of transmission lines.

At a given bus i, the phasor source current into the bus is given by Kirchhoff's Current Law as:

$$\overline{I}_i(t) = \sum_{j=o}^{N} [\overline{E}_i(t) - \overline{E}_j(t)]\overline{Y}^{ij}(t) \tag{6.2.1}$$

where

$$\overline{Y}^{ij} = G^{ij} + jB^{ij} \tag{6.2.2}$$

The net power and reactive volt-ampere at the ith bus is

$$\overline{S}_i(t) = \overline{E}_i^*(t)\overline{I}_i(t) \tag{6.2.3}$$

$$\overline{S}_i(t) = P_i(t) - jQ_i(t) \tag{6.2.4}$$

where $E_i^*(t)$ represents the conjugate phasor voltage. Let:

$$\overline{E}_i(t) = E_i(t)e^{j\theta_i} \tag{6.2.5}$$

$$\overline{E}_i(t) = E_{di}(t) + jE_{q_i}(t) \tag{6.2.6}$$

Then (6.2.3) reduces to:

$$\overline{S}_i(t) = \sum_{j=o}^{N} \overline{E}_i^*(t)\overline{E}_i(t)\overline{Y}^{ij} - \overline{E}_i^*(t) \sum_{\substack{j=o \\ j\neq i}}^{N} \overline{E}_j(t)\overline{Y}^{ij} \tag{6.2.7}$$

substituting (6.2.2) and (6.2.6) in (6.2.7) and separating real and imaginery parts of $\overline{S}_i$ to obtain P_i and Q_i as given by (6.2.4) one obtains:

$$P_i(t) = E_i^2(t)G_i - E_{di}(t) \sum_{j=o}^{N} [E_{d_j}(t)G^{ij} - E_{q_j}(t)B^{ij}]$$

$$- E_{q_i}(t) \sum_{\substack{j=o \\ j\neq i}}^{N} [E_{d_j}(t)B^{ij} + E_{q_j}(t)G^{ij}] \tag{6.2.8}$$

$$-Q_i(t) = E_i^2(t)B_i + E_{q_i}(t) \sum_{\substack{j=o \\ j\neq i}}^{N} [E_{d_j}(t)G^{ij} - E_{q_j}(t)B^{ij}]$$

$$- E_{d_i}(t) \sum_{\substack{j=o \\ j\neq i}}^{N} [E_{d_j}(t)B^{ij} + E_{q_j}(t)G^{ij}] \tag{6.2.9}$$

where

$$G_i = \sum_{\substack{j=o \\ j\neq i}}^{N} G^{ij} \tag{6.2.10}$$

$$B_i = \sum_{\substack{j=o \\ j\neq i}}^{N} B^{ij} \tag{6.2.11}$$

Note that if the $\overline{E}_i$'s are the phasor voltage to neutral then it is obvious that $E_{d_o} = E_{q_o} = 0$. Hence (6.2.8) and (6.2.9) are rewritten as:

$$P_i(t) = E_i^{\,2}(t)G_i - E_{d_i}(t)\sum_{\substack{j=1 \\ j\neq i}}^{N}[E_{d_j}(t)G^{ij} - E_{q_j}(t)B^{ij}]$$

$$- E_{q_i}(t)\sum_{\substack{j=1 \\ j\neq i}}^{N}[E_{d_j}(t)B^{ij} + E_{q_j}(t)G^{ij}]$$

$$i = 1,\ldots,N \tag{6.2.12}$$

$$-Q_i(t) = E_i^{\,2}(t)B_i + E_{q_i}(t)\sum_{\substack{j\neq 1 \\ j\neq i}}^{N}[E_{d_j}(t)G^{ij} - E_{q_j}(t)B^{ij}]$$

$$- E_{d_i}(t)\sum_{\substack{j=1 \\ j\neq i}}^{N}[E_{d_j}(t)B^{ij} + E_{q_j}(t)G^{ij}]$$

$$i = 1,\ldots,N \tag{6.2.13}$$

Note that

$$E_i^{\,2} = E_{d_i}^{\,2}(t) + E_{q_i}^{\,2}(t) \qquad i = 1,\ldots,N \tag{6.2.14}$$

The equations (6.2.12) and (6.2.13) are the load flow equations. Each bus is characterized by four variables $P_i(t)$, $Q_i(t)$, $E_{d_i}(t)$ and $E_{q_i}(t)$. In a normal load flow study, two of the four variables are specified and the others must be found. Depending upon which variables are specified, the buses can be divided into three types [60]:

1. Generator bus with P and E specified, Q and the phase angle $[\tan^{-1} \frac{E_q}{E_d}]$ unknown.

2. Load bus with P and Q specified, E_d and E_q being the unknowns.

3. Slack bus with E_d and E_q specified, P and Q unknowns. For convenience this shall be the node N_g and $E_q(t)$ is taken as zero. Since

the slack bus is taken as a generator N_g node, this means that the number of the unknowns is reduced by one. We may assume that E is not specified at the (N_g-1) bus.

In the economy dispatch problem, the active power generation P at the generator buses are sought. These generations are obtained such that maximum economy is achieved. This requires minimizing the operating costs at the thermal plants. Thus the problem is:

$$\underset{P_{s_i}(t)}{\mathrm{Min}} \int_o^{T_f} \{ \sum_{i=N_h+1}^{N_g} [\alpha_i + \beta_i P_{s_i}(t) + \gamma_i P_{s_i}{}^2(t)]\} dt \qquad (6.2.15)$$

while satisfying the load flow equations. Note that now the powers P_i's are no longer specified for the generator buses in these equations.

Another important objective in the power system's operation is its reliability. To improve the system reliability during operation, it is necessary to ensure that the reactive generations are minimally proportional between the system generators [32,64]. To achieve this it is suggested that the schedules obtained must

$$\underset{Q_{G_i}(t)}{\mathrm{Min}} \int_o^{T_f} [\sum_{i=1}^{N_g} \sum_{j=1}^{N_g} Q_i(t) K_{ij} Q_j(t)] dt \qquad (6.2.16)$$

where the K_{ij}'s are assumed to be known weighting coefficients.

It is assumed here, that both the economy requirement and the reliability requirement are of equal importance. Thus one is required to:

$$\underset{P_{s_i}(t), Q_{G_i}(t)}{\mathrm{Min}} \int_o^{T_f} \{ \sum_{i=N_h+1}^{N_g} [\beta_i P_{s_i}(t) + \gamma_i P_{s_i}{}^2(t)] + \sum_{i=1}^{N_g} \sum_{j=1}^{N_g} Q_i(t) K_{ij} Q_j(t)\} dt \qquad (6.2.17)$$

while satisfying the load flow equations (6.2.12) and (6.2.13).

There are several inequality constraints that must be satisfied at the optimum for a valid solution. These are given by:

$$P_i^2(t) + Q_i^2(t) \leq S_i^{2^M} \qquad i = 1,\ldots,N_g \qquad (6.2.18)$$

$$Q_i^m \leq Q_i(t) \leq Q_i^M \qquad i = 1,\ldots,N_g \qquad (6.2.19)$$

$$P_i^m \leq P_i(t) \leq P_i^M \qquad i = 1,\ldots,N_g \qquad (6.2.20)$$

Furthermore, since E at a generator bus is specified then one requires

$$E_{d_i}^2(t) + E_{q_i}^2(t) = E_i^2(t) \qquad i = 1,\ldots,N_g-1 \qquad (6.2.21)$$

where $E_i(t)$ is assumed to be known.

The hydro-plants active power generation is assumed to vary with the rate of water discharge as:

$$P_{h_i}(t) + A_i(t)q_i(t) + B_{W_i}q_i(t)Q_{W_i}(t) + C_iq_i^2(t) = 0$$
$$i = 1,\ldots,N_h \qquad (6.2.22)$$

with

$$q_i(t) = \dot{Q}_{W_i}(t) \qquad i = 1,\ldots,N_h \qquad (6.2.23)$$

Moreover, the volume of water discharge at any hydro-plant is a prespecified constant:

$$\int_0^{T_f} q_i(t)dt = b_i \qquad i = 1,\ldots,N_h \qquad (6.2.24)$$

6.3 A Minimum Norm Formulation

The problem stated in the previous section can be formulated as a minimum norm problem as follows: An augmented cost functional J_0 can be obtained as:

$$J_o(o) = \sum_{i=1}^{10} J_{o_i}(.) \tag{6.3.1}$$

$$J_{o_1}(.) = \int_o^{T_f} [\sum_{i=1}^{N} \lambda_{p_i}(t)[-P_i(t) + E_{d_i}^{\ 2}(t)G_i + E_{q_i}^{\ 2}(t)G_i$$

$$- E_{d_i}(t) \sum_{\substack{j=1 \\ j\neq i}}^{N} [E_{d_j}(t)G^{ij} - E_{q_j}(t)B^{ij}]$$

$$- E_{q_i}(t) \sum_{\substack{j=1 \\ j\neq i}}^{N} [E_{d_j}(t)B^{ij} + E_{q_j}(t)G^{ij}]]]dt \tag{6.3.2}$$

$$J_{o_2}(.) = \int_o^{T_f} \sum_{i=1}^{N} \lambda_{q_i}(t)[Q_i(t) + E_{d_i}^{\ 2}(t)B_i + E_{q_i}^{\ 2}(t)B_i$$

$$+ E_{q_i}(t) \sum_{\substack{j=1 \\ j\neq i}}^{N} [E_{d_j}(t)G^{ij} - E_{q_j}(t)B^{ij}]$$

$$- E_{d_i}(t) \sum_{\substack{j=1 \\ j\neq i}}^{N} [E_{d_j}(t)B^{ij} + E_{q_i}(t)G^{ij}]]dt \tag{6.3.3}$$

$$J_{o_3}(.) = \int_o^{T_f} \sum_{i=1}^{N_g-2} \lambda_{e_i}(t)[E_{d_i}^{\ 2}(t) + E_{q_i}^{\ 2}(t)]dt \tag{6.3.4}$$

$$J_{o_4}(.) = \int_o^{T_f} \sum_{i=1}^{N_h} \{n_i(t)[P_{h_i}(t) + A_i(t)q_i(t) + B_{W_i}q_i(t)$$

$$Q_{W_i}(t) + C_i q_i^{\ 2}(t)] + m_i(t)q_i(t) + \hat{m}_i(t)Q_{W_i}(t)\}dt \tag{6.3.5}$$

$$J_{o_5}(.) = \int_0^{T_f} \{ \sum_{i=N_h+1}^{N_g} [\beta_i P_{s_i}(t) + \gamma_i P_{s_i}^2(t)]$$

$$+ \sum_{i=1}^{N_g} \sum_{j=1}^{N_g} Q_i(t) K_{ij} Q_j(t) \} dt \tag{6.3.6}$$

$$J_{o_6}(.) = \int_0^{T_f} \sum_{i=1}^{N_g} M_i(t)[P_i^2(t) + Q_i^2(t) - S_i^{2^M}] dt \tag{6.3.7}$$

$$J_{o_7}(.) = \int_0^{T_f} \sum_{i=1}^{N_g} \ell_i(t)[P_i^m - P_i(t)] dt \tag{6.3.8}$$

$$J_{o_8}(.) = \int_0^{T_f} \sum_{i=1}^{N_g} \ell'_i(t)[P_i(t) - P_i^M] dt \tag{6.3.9}$$

$$J_{o_9}(.) = \int_0^{T_f} \sum_{i=1}^{N_g} e_i(t)[Q_i^m - Q_i(t)] dt \tag{6.3.10}$$

$$J_{o_{10}}(.) = \int_0^{T_f} \sum_{i=1}^{N_g} e'_i(t)[Q_i(t) - Q_i^M] dt \tag{6.3.11}$$

Here J_{o_1} is obtained from (6.2.12) and J_{o_2} is obtained from (6.2.13). Also J_{o_3} corresponds to (6.2.21), J_{o_4} to (6.2.22) and (6.2.23). J_{o_5} is the original cost functional of (6.2.17). The inequality constraints (6.2.18) through (6.2.20) are included using the Kuhn-Tucker theorem [59] by considering $J_{o_6}(.)$ through $J_{o_{10}}(.)$ so that the following exclusion equations must be satisfied at the optimum.

$$M_i(t)[P_i^2(t) + Q_i^2(t) - S_i^{2^M}] = 0 \tag{6.3.12}$$

$$\ell_i(t)[P_i^m - P_i(t)] = 0 \tag{6.3.13}$$

$$\ell'_i(t)[P_i(t) - P_i^M] = 0 \tag{6.3.14}$$

$$e_i(t)[Q_i^m - Q_i(t)] = 0 \qquad (6.3.15)$$

$$e_i'(t)[Q_i(t) - Q_i^M] = 0 \qquad (6.3.16)$$

for $i = 1,\ldots,N_g$ $t \in [0,T_f]$. Moreover, $\lambda_{p_i}(t)$, $\lambda_{q_i}(t)$, $\lambda_{e_i}(t)$, $n_i(t)$ and $m_i(t)$ are to be determined such that the corresponding equality constraints are satisfied.

The augmented cost functional $J_o(.)$ of (6.3.1) can also be expressed as:

$$J_o(.) = J_{o_d}(.) + J_{o_p}(.) + J_{o_Q}(.) + J_{o_E}(.) + J_{o_W}(.) \qquad (6.3.17)$$

where

$$J_{o_d}(.) = \int_0^{T_f} \sum_{i=N_g+1}^{N} [-\lambda_{p_i}(t)P_i(t) + \lambda_{q_i}(t)Q_i(t)]$$

$$+ [\lambda_{p_{N_g}}(t)G_{N_g} + \lambda_{q_{N_g}}(t)B_N]$$

$$[E_{d_{N_g}}{}^2(t) + E_{q_{N_g}}{}^2(t)] + \sum_{i=1}^{N_g}$$

$$[-M_i(t)S_i^{2^M} + \ell_i(t)P_i^m - \ell_i'(t)P_i^M$$

$$+ e_i(t)Q_i^m - e_i'(t)Q_i^M]dt \qquad (6.3.18)$$

Note that $J_{o_d}(.)$ is explicitly independent of the control variables. The control variables here are $P_i(t)$ and $Q_i(t)[i = 1,\ldots,N_g]$, $E_{d_i}(t)$, $E_{q_i}(t)$ $[i = 1,\ldots,N]$ and $q_i(t)$,$Q_{W_i}(t)[i = 1,\ldots,N_h]$. And

$$J_{o_p}(.) = \int_0^{T_f} [\sum_{i=1}^{N_h} [n_i(t) + \ell_i'(t) - \ell_i(t) - \lambda_{p_i}(t)]P_i(t)$$

$$+ \sum_{i=N_h+1}^{N_g} [\beta_i + \ell_i'(t) - \ell_i(t) - \lambda_{p_i}(t)]P_i(t)$$

$$+ \sum_{i=1}^{N_h} M_i(t)P_i^2(t) + \sum_{i=N_h+1}^{N_g} (M_i(t) + \gamma_i)$$

$$P_i^2(t)]dt \tag{6.3.19}$$

$$J_{o_Q}(.) = \int_o^{T_f} [\sum_{i=1}^{N_g} [e_i'(t) - e_i(t) + \lambda_{q_i}(t)]Q_i(t)$$

$$+ \sum_{i=1}^{N_g} M_i(t)Q_i^2(t) + \sum_{i=1}^{N_g} \sum_{j=1}^{N_g} Q_i(t)K_{ij}Q_j(t)]dt \tag{6.3.20}$$

$$J_{o_E}(.) = \int_o^{T_f} [\sum_{i=1}^{N_g-2} [\lambda_{p_i}(t)G_i + \lambda_{q_i}(t)B_i + \lambda_{e_i}(t)]E_{d_i}^2(t)$$

$$+ \sum_{i=1}^{N_g-2} [\lambda_{p_i}(t)G_i + \lambda_{q_i}(t)B_i + \lambda_{e_i}(t)]E_{q_i}^2(t)$$

$$+ \sum_{i=N_g-1}^{N} [\lambda_{p_i}(t)G_i + \lambda_{q_i}(t)B_i]E_{d_i}^2(t)$$

$$+ \sum_{i=N_g-1}^{N} [\lambda_{p_i}(t)G_i + \lambda_{q_i}(t)B_i]E_{q_i}^2(t)$$

$$- \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} E_{d_i}(t)\{\lambda_{p_i}(t)G^{ij} + \lambda_{q_i}(t)B^{ij}\}E_{d_j}(t)$$

$$- \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} E_{q_i}(t)\{\lambda_{p_i}(t)G^{ij} + \lambda_{q_i}(t)B^{ij}\}E_{q_j}(t)$$

$$- \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} E_{q_i}(t)\{\lambda_{p_i}(t)B^{ij} - \lambda_{q_i}(t)G^{ij}\}E_{d_j}(t)$$

$$+ \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} E_{d_i}(t)\{\lambda_{p_i}(t)B^{ij} - \lambda_{q_i}(t)G^{ij}\}E_{q_j}(t)]dt \tag{6.3.21}$$

Let

$$a_{ij}(t) = -[\lambda_{p_i}(t)G^{ij} + \lambda_{q_i}(t)B^{ij}]$$

$$i,j = 1,\ldots,N, i \neq 1 \tag{6.3.22}$$

$$b_{ij}(t) = -[\lambda_{p_i}(t)B^{ij} - \lambda_{q_i}(t)G^{ij}]$$

$$i,j = 1,\ldots,N, i \neq j \tag{6.3.23}$$

$$a_{ii}(t) = [\lambda_{p_i}(t)G_i + \lambda_{q_i}(t)B_i + \lambda_{e_i}(t)]$$

$$i = 1,\ldots,N_g-2 \tag{6.3.24}$$

$$a_{ii}(t) = \lambda_{p_i}(t)G_i + \lambda_{q_i}(t)B_i$$

$$i = N_g-1,\ldots,N \tag{6.3.25}$$

Then (6.3.21) reduces to:

$$J_{o_E} = \int_o^{T_f} [\sum_{\substack{i=1 \\ i \neq N_g}}^{N} a_{ii}(t)[E_{d_i}{}^2(t) + E_{q_i}{}^2(t)]$$

$$+ \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} E_{d_i}(t)a_{ij}(t)E_{d_j}(t)$$

$$+ \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} E_{q_i}(t)a_{ij}(t)E_{q_j}(t)$$

$$+ \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} E_{q_i}(t) b_{ij}(t) E_{d_j}(t)$$

$$- \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} E_{d_i}(t) b_{ij}(t) E_{q_j}(t)]dt \qquad (6.3.26)$$

Since $E_{d_{N_g}}(t)$ and $E_{q_{N_g}}(t)$ are specified (this is the slack bus) then the above expression can be reduced to

$$J_{o_E}(.) = \int_{o}^{T_f} [\sum_{\substack{i=1 \\ i \neq N_g}}^{N} a_{ii}(t)[E_{d_i}{}^2(t) + E_{q_i}{}^2(t)]$$

$$+ \sum_{\substack{i=1 \\ i \neq N_g}}^{N} \sum_{\substack{j=1 \\ j \neq i, j \neq N_g}}^{N} E_{d_i}(t) a_{ij}(t) E_{d_j}(t)$$

$$+ \sum_{\substack{i=1 \\ i \neq N_g}}^{N} \sum_{\substack{j=1 \\ j \neq i, j \neq N_g}}^{N} E_{q_i}(t) a_{ij}(t) E_{q_j}(t)$$

$$+ \sum_{\substack{i=1 \\ i \neq N_g}}^{N} \sum_{\substack{j=1 \\ j \neq i, j \neq N_g}}^{N} E_{q_i}(t) B_{ij}(t) E_{d_j}(t)$$

$$- \sum_{\substack{i=1 \\ i \neq N_g}}^{N} \sum_{\substack{j=1 \\ j \neq i, j \neq N_g}}^{N} E_{d_i}(t) b_{ij}(t) E_{q_j}(t)$$

$$+ \sum_{i=1, i \neq N_g}^{N} [a_{N_g,i}(t) + a_{i,N_g}(t)] E_{d_{N_g}}(t) E_{d_i}(t)$$

$$+ \sum_{\substack{i=1 \\ i \neq N_g}}^{N} [a_{N_g,i}(t), a_{i,N_g}(t)] E_{q_{N_g}}(t) E_{q_i}(t)$$

$$+ \sum_{\substack{i=1 \\ i \neq N_g}}^{N} [b_{N_g,i}(t) + b_{i,N_g}(t)] E_{q_{N_g}}(t) E_{d_i}(t)$$

$$- \sum_{\substack{i=1 \\ i \neq N_g}}^{N} [b_{N_g,i}(t) + b_{i,N_g}(t)] E_{d_{N_g}}(t) E_{q_i}(t)]dt \tag{6.3.27}$$

With the assumption of zero phase angle at the slack bus:

$$E_{q_{N_g}}(t) = 0 \tag{6.3.28}$$

thus

$$J_{o_E}(.) = \int_0^{T_f} [\sum_{\substack{i=1 \\ i \neq N_g}}^{N} \sum_{\substack{j=1 \\ j \neq N_g}}^{N} E_{d_i}(t) a_{ij}(t) E_{d_j}(t)$$

$$+ \sum_{\substack{i=1 \\ i \neq N_g}}^{N} \sum_{\substack{j=1 \\ j \neq N_g}}^{N} E_{q_i}(t) a_{ij}(t) E_{q_j}(t)$$

$$+ \sum_{\substack{i=1 \\ i \neq N_g}}^{N} \sum_{\substack{j=1 \\ j \neq i, j \neq N_g}}^{N} E_{q_i}(t) b_{ij}(t) E_{d_j}(t)$$

$$- \sum_{\substack{i=1 \\ i \neq N_g}}^{N} \sum_{\substack{j=1 \\ j \neq i, j \neq N_g}}^{N} E_{d_i}(t) b_{ij}(t) E_{q_j}(t)$$

$$+ \sum_{\substack{i=1 \\ i \neq N_g}}^{N} [a_{N_g,i}(t) + a_{i,N_g}(t)] E_{d_{N_g}}(t) E_{d_i}(t)$$

$$- \sum_{\substack{i=1 \\ i \neq N_g}}^{N} [b_{N_g,i}(t) + b_{i,N_g}(t)] E_{d_{N_g}}(t) E_{q_i}(t)]dt \tag{6.3.29}$$

And

$$J_{o_W}(.) = \int_o^{T_f} [\sum_{i=1}^{N_h} (n_i(t)A_i(t) + m_i(t))q_i(t)$$

$$+ \sum_{i=1}^{N_h} \dot{m}_i(t)Q_{W_i}(t) + \sum_{i=1}^{N_h} C_i n_i(t)q_i^2(t)$$

$$- \sum_{i=1}^{N_h} \frac{B_{W_i}\dot{n}_i(t)}{2} Q_{W_i}^2(t)]dt \qquad (6.3.30)$$

Define the control vector as:

$$\underline{u}(t) = \text{col.}[\underline{P}(t),\underline{Q}(t),\underline{E}(t),\underline{W}(t)] \qquad (6.3.31)$$

with

$$\underline{P}(t) = \text{col.}[\underline{P}_h(t),\underline{P}_s(t)] \qquad (6.3.32)$$

$$\underline{P}_h(t) = \text{col}[P_1(t),\ldots,P_{N_h}(t)] \qquad (6.3.33)$$

$$\underline{P}_s(t) = \text{col.}[\underline{P}_{N_h+1}(t),\ldots,P_{N_g}(t)] \qquad (6.3.34)$$

$$\underline{Q}(t) = \text{col.}[Q_1(t),\ldots,Q_{N_g}(t)] \qquad (6.3.35)$$

$$\underline{E}(t) = \text{col.}[\underline{E}_d(t),\underline{E}_q(t)] \qquad (6.3.36)$$

$$\underline{E}_d(t) = \text{col.}[\underline{E}_{d_1}(t),\ldots,E_{d_{N_g-1}}(t),E_{d_{N_g+1}}(t),E_{d_N}(t)] \qquad (6.3.37)$$

$$\underline{E}_q(t) = \text{col.}[E_{q_1}(t),\ldots,E_{q_{N_g-1}}(t),E_{q_{N_g+1}}(t),E_{q_N}(t)] \qquad (6.3.38)$$

$$\underline{W}(t) = \text{col.}[\underline{W}_1(t),\ldots,\underline{W}_{N_h}(t)] \qquad (6.3.39)$$

$$\underline{W}_i(t) = \text{col.}[q_i(t),Q_{W_i}(t)] \qquad (6.3.40)$$

The control is a $2[n+N_h+N_g]x1$ column vector function as can be seen by inspecting (6.3.31) through (6.3.40).

Define the auxilliary vector $\underline{L}(t)$ as:

$$\underline{L}(t) = \text{col.}[\underline{L}_P(t),\underline{L}_Q(t),\underline{L}_E(t),\underline{L}_W(t)] \tag{6.3.41}$$

$$\underline{L}_P(t) = \text{col.}[\underline{L}_{P_h}(t),\underline{L}_{P_s}(t)] \tag{6.3.42}$$

$$\underline{L}_Q(t) = \text{col.}[L_{Q_1}(t),\ldots,L_{Q_{N_g}}(t)] \tag{6.4.43}$$

$$\underline{L}_{P_h}(t) = \text{col.}[L_{P_1}(t),\ldots,L_{P_{N_h}}(t)] \tag{6.4.44}$$

$$\underline{L}_{P_s}(t) = \text{col.}[L_{P_{N_h+1}}(t),\ldots,L_{P_{N_g}}(t)] \tag{6.3.45}$$

$$\underline{L}_E(t) = \text{col.}[\underline{L}_{E_d}(t),\underline{L}_{E_q}(t)] \tag{6.4.46}$$

$$\underline{L}_{E_d}(t) = \text{col.}[L_{E_{d_1}}(t),\ldots,L_{E_{d_{N_g-1}}}(t),L_{E_{d_{N_g+1}}}(t),L_{E_{d_N}}(t)] \tag{6.3.47}$$

$$\underline{L}_{E_q}(t) = \text{col.}[L_{E_{q_1}}(t),\ldots,L_{E_{q_{N_g-1}}}(t),L_{E_{q_{N_g+1}}}(t),\underline{L}_{E_{q_N}}(t)] \tag{6.3.48}$$

$$\underline{L}_W(t) = \text{col.}[\underline{L}_{W_1}(t),\ldots,\underline{L}_{W_{N_h}}(t)] \tag{6.3.49}$$

with

$$L_{P_i}(t) = [n_i(t) + \ell_i'(t) - \ell_i(t) - \lambda_{p_i}(t)] \qquad i = 1,\ldots,N_h \tag{6.3.50}$$

$$L_{P_i}(t) = [\beta_i + \ell_i'(t) - \ell_i(t) - \lambda_{p_i}(t)] \qquad i = N_h+1,\ldots,N_g \tag{6.3.51}$$

$$L_{Q_i}(t) = [e_i'(t) - e_i(t) + \lambda_{q_i}(t)] \qquad i = 1,\ldots,N_g \tag{6.3.52}$$

$$L_{E_{d_i}}(t) = [a_{N_g,i}(t) + a_{i,N_g}(t)]E_{d_{N_g}}(t) \qquad i = 1,\ldots,N,\ i \neq N_g \tag{6.3.53}$$

$$L_{E_{q_i}}(t) = -[b_{N_g,i}(t) + b_{i,N_g}(t)]E_{d_{N_g}}(t) \qquad i = 1,\ldots,N \quad i \neq N_g \tag{6.3.54}$$

$$\underline{L}_{W_i}(t) = \text{col.}[(n_i(t)A_i(t) + m_i(t)),\dot{m}_i(t)] \qquad i = 1,\ldots,N_h \tag{6.3.55}$$

Let the square matrix $\underline{B}(t)$ be given by:

$$\underline{B}(t) = \text{diag}[\underline{B}_p(t),\underline{B}_Q(t),\underline{B}_E(t),\underline{B}_W(t)] \tag{6.3.56}$$

with

$$\underline{B}_p(t) = \text{diag}[\underline{B}_{p_h}(t),\underline{B}_{p_s}(t)] \tag{6.3.57}$$

$$\underline{B}_{p_h}(t) = \text{diag}[M_i(t)] \qquad i = 1,\ldots,N_h \tag{6.3.58}$$

$$\underline{B}_{p_s}(t) = \text{diag}[B_{p_i}(t)] \qquad i = N_h+1,\ldots,N_g \tag{6.3.59}$$

$$B_{p_i}(t) = M_i(t) + \gamma_i \tag{6.3.60}$$

$$\underline{B}_Q(t) = (K'_{ij}(t)) \tag{6.3.61}$$

$$K'_{ij} = K_{ij} \qquad i \neq j \quad i,j = 1,\ldots,N_g \tag{6.3.62}$$

$$K'_{ii}(t) = M_i(t) + K_{ii} \qquad i = 1,\ldots,N_g \tag{6.3.63}$$

$$\underline{B}_W(t) = \text{diag}[\underline{B}_{W_i}(t)] \qquad i = 1,\ldots,N_h \tag{6.3.64}$$

$$\underline{B}_{W_i}(t) = \text{diag}[C_i n_i(t), -\frac{B_i\dot{n}_i(t)}{2}] \qquad i = 1,\ldots,N_h \tag{6.3.65}$$

Let

$$\underline{B}_{\underline{E}_o}(t) = \left[\begin{array}{cccc:cccc}
a_{11}(t) & a_{12}(t) & \cdots & a_{1N}(t) & 0 & -b_{12}(t) & -b_{13}(t) \cdots & -b_{1N}(t) \\
a_{21}(t) & a_{22}(t) & \cdots & a_{2N}(t) & -b_{21}(t) & 0 & -b_{23}(t) \cdots & -b_{2N}(t) \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{N1}(t) & a_{N2}(t) & \cdots & a_{NN}(t) & -b_{N1}(t) & -b_{N2}(t) & -b_{N3}(t) \cdots & 0 \\
\hdashline
0 & b_{12}(t) & b_{13}(t) \cdots & b_{1N}(t) & a_{11}(t) & a_{12}(t) & \cdots & a_{1N}(t) \\
b_{21}(t) & 0 & b_{23}(t) \cdots & b_{2N}(t) & a_{21}(t) & a_{22}(t) & \cdots & a_{2N}(t) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
b_{N1}(t) & b_{N2}(t) & b_{N3}(t) \cdots & 0 & a_{N1}(t) & a_{N2}(t) & \cdots & a_{NN}(t)
\end{array}\right]$$

or $\underline{B}_{\underline{E}_o}(t) = (C_{ij}(t))_{2(N-1)x2(N-1)}$ $\quad i,j \neq N_g$

$i,j \neq N+N_g$ (6.3.66)

Then (6.3.29) reduces to

$$J_{o_E}(.) = \int_0^{T_f} [\underline{E}^T(t)\underline{B}_{\underline{E}_o}(t)\underline{E}(t) + \underline{L}_{\underline{E}}^T(t)\underline{E}(t)] \qquad (6.3.67)$$

Note that $\underline{B}_{\underline{E}_o}(t)$ is non symmetric. However, one can replace $\underline{B}_{\underline{E}_o}(t)$ by the symmetric matrix $\underline{B}_{\underline{E}}(t)$ such that

$$\underline{E}^T(t)\underline{B}_{\underline{E}_o}(t)\underline{E}(t) = \underline{E}^T\underline{B}_{\underline{E}}(t)\underline{E}(t) \qquad (6.3.68)$$

where

$$\underline{B}_{\underline{E}}(t) = (b_{e_{ij}}(t))_{2(N-1)x2(N-1)} \qquad (6.3.69)$$

with

$$b_{e_{ij}}(t) = b_{e_{ji}}(t) \qquad (6.3.70)$$

$$b_{e_{ij}}(t) = \tfrac{1}{2}(C_{ij}(t) + C_{ji}(t))) \quad i,j = 1,\ldots,2N$$

$$j,j \neq N_g \text{ and } N+N_g \qquad (6.3.71)$$

Using the above definitions, the augmented cost functional $J_o(.)$ of (6.3.17) reduces to:

$$J_1(.) = \int_0^{T_f} [\underline{L}^T(t)\underline{u}(t) + \underline{u}^T(t)\underline{B}(t)\underline{u}(t)]dt \tag{6.3.72}$$

Note that the terms explicitly independent of the control $\underline{u}(t)$ are dropped in (6.3.72)

Let

$$V^T(t) = L^T(t)B^{-1}(t) \tag{6.3.73}$$

then the cost functional of (6.3.72) becomes:

$$J_2(.) = \int_0^{T_f} [\underline{u}(t) + \frac{\underline{V}(t)}{2}]^T\underline{B}(t)[\underline{u}(t) + \frac{\underline{V}(t)}{2}]dt \tag{6.3.74}$$

Here $-\frac{V^T(t)}{2}\underline{B}(t)\frac{\underline{V}(t)}{2}$ was dropped since it is explicitly independent of $\underline{u}(t)$.

Thus the problem is now reduced to that of minimizing (6.3.74) subject to satisfying (6.2.24) which is:

$$b_i = \int_0^{T_f} q_i(t)dt \qquad i = 1,\ldots,N_h \tag{6.3.75}$$

Define the N_hx1 column vector:

$$\underline{b} = \text{col.}[b_1,\ldots,b_{N_h}] \tag{6.3.76}$$

and the N_hx$[2(N+N_h+N_g)]$ matrix $\underline{K}^T$ as:

$$\underline{K}^T = [\underline{0}_P, \underline{0}_Q, \underline{0}_E, \underline{K}'^T_W] \tag{6.3.77}$$

with $\underline{0}_P$ being the N_hxN_g matrix whose elements are all zero. $\underline{0}_Q$ and $\underline{0}_E$ are N_hx(N_g) and N_hx$(2(N-1)$ zero matrices respectively. And $\underline{K}'^T_W$ being the N_hx$2N_h$ matrix given by:

$$\underline{K}'^T_W = [\underline{K}'^T_{W_1},\ldots,\underline{K}'^T_{W_{N_h}}] \tag{6.3.78}$$

The $\underline{K}'^T_{W_i}$'s being N_hx2 matrices with zero elements everywhere except at the

1st column and ith row location where the element is one. Thus (6.3.75) reduces to:

$$\underline{b} = \int_0^{T_f} \underline{K}^T \underline{u}(s)ds \qquad (6.3.79)$$

The control vector $\underline{u}(t)$ is considered an element of the Hilbert space $L_{2,B}^{2(n+N_h+N_g)}[0,T_f]$ of the $2(N+N_h+N_g)$ vector valued square integrable functions defined on $[0,T_f]$ endowed with the inner product definition:

$$<\underline{V}(t),\underline{u}(t)> = \int_0^{T_f} \underline{V}^T(t)\underline{B}(t)\underline{u}(t)dt \qquad (6.3.80)$$

for every $\underline{V}(t)$ and $\underline{u}(t)$ in $L_{2,B}^{2(N_h+N_g+N)}[0,T_f]$, provided that $\underline{B}(t)$ is positive definite.

The given vector $\underline{b}$ is considered an element of the Real space R^{N_h} with the Euclidean inner product definition.

$$<\underline{X},\underline{Y}> = \underline{X}^T\underline{Y} \qquad (6.3.81)$$

for every $\underline{X}$ and $\underline{Y}$ in R^{N_h}:

Equation (6.3.79) defines a bounded linear transformation $T: L_{2,\underline{B}}^{2[N_h+N_g+N]}[0,T_f] \to R^{N_h}$. This can be expressed as:

$$\underline{b} = T[\underline{u}(t)] \qquad (6.3.82)$$

and the cost functional given by (6.3.74) reduces to:

$$J_2[\underline{u}(t)] = ||\underline{u}(t) + \frac{\underline{V}(t)}{2}||^2 \qquad (6.3.83)$$

Finally it is necessary only to minimize:

$$J[\underline{u}(t)] = ||\ (t) + \frac{\underline{V}(t)}{2}|| \qquad (6.3.84)$$

subject to $\underline{b} = T[\underline{u}(t)]$ (6.3.85)

for a given $\underline{b}$ in R^{N_h}

6.4 The optimal solution

The optimal solution to the problem formulated in the previous section using the results of Chapter 2 is:

$$\underline{u}_\xi = T^\dagger[\underline{b} + T(\frac{\underline{V}(t)}{2})] - \frac{\underline{V}(t)}{2} \tag{6.4.1}$$

where $T^\dagger$ is obtained as follows:

T^*, the adjoint of T, is obtained using the identity:

$$<\underline{\xi},\underline{Tu}>_{R^{N_h}} = <\underline{T^*\xi},\underline{u}>_{L_{2,B}^{2(N_h+N_g+N)}[0,T_f]} \tag{6.4.2}$$

Let

$$\underline{\xi} = \text{col.}[\xi_1,\ldots,\xi_{N_h}] \tag{6.4.3}$$

$$\underline{T^*\xi} = \text{col.}[\underline{T}_p,\underline{T}_Q,\underline{T}_E,\underline{T}_W] \tag{6.4.4}$$

where $\underline{T}_p$, $\underline{T}_Q$ and $\underline{T}_E$ are of the same dimension as $\underline{P}$,$\underline{Q}$ and $\underline{E}$ respectively.

$$\underline{T}_W = \text{col.}[\underline{T}_{W_1},\ldots,\underline{T}_{W_{N_h}}] \tag{6.4.5}$$

$$\underline{T}_{W_i} = \text{col.}[T_{W_{i_q}},T_{W_{i_{Q_W}}}] \qquad i = 1,\ldots,N_h \tag{6.4.6}$$

and

$$\underline{\phi}_i = \text{col.}[\xi_i,0] \qquad i = 1,\ldots,N_h \tag{6.4.7}$$

Then in R^{N_h} one obtains:

$$<\underline{\xi},\underline{Tu}> = \int_0^{T_f} \sum_{i=1}^{N_h} \underline{\phi}_i{}^T \underline{W}_i dt \tag{6.4.8}$$

and in $L_{2,B}^{2(N_h+N_g+N)}[0,T_f]$:

$$<\underline{T^*\xi},\underline{u}> = \int_0^{T_f} [\underline{T}_p,\underline{B}_p(t)\underline{P}(t) + \underline{T}_Q\underline{B}_Q(t)\underline{Q}(t) + \underline{T}_E\underline{B}_E(t)\underline{E}(t)$$
$$+ \sum_{i=1}^{N_h} \underline{T}_{W_i}\underline{B}_{W_i}(t)\underline{W}_i(t)]dt \tag{6.4.9}$$

Thus the equality of (6.4.2) reduces to:

$$\int_0^{T_f} \sum_{i=1}^{N_h} \underline{\phi}_i^T \underline{W}_i dt = \int_0^{T_f} [\underline{T}_p \underline{B}_p(t) \underline{P}(t) + \underline{T}_Q \underline{B}_Q(t) \underline{Q}(t)$$

$$+ \underline{T}_E \underline{B}_E(t) \underline{E}(t) + \sum_{i=1}^{N_h} \underline{T}_{W_i} \underline{B}_{W_i}(t) \underline{W}_i(t)]dt \tag{6.4.10}$$

which is satisfied for

$$\underline{T}_p = \underline{0} \tag{6.4.11}$$

$$\underline{T}_Q = \underline{0} \tag{6.4.12}$$

$$\underline{T}_E = \underline{0} \tag{6.4.13}$$

$$\underline{T}_{W_i} = \phi_i^T \underline{B}_{W_i}^{-1}(t) \qquad i = 1,\ldots,N_h$$

or

$$\underline{T}_{W_i} = [\frac{\xi_i}{C_i n_i(t)}, 0] \qquad i = 1,\ldots,N_h \tag{6.4.14}$$

This completely defines $\underline{T^* \xi}$ as given by (6.4.4).

The operator J is evaluated as

$$J[\underline{\xi}] = T[T^* \xi] \tag{6.4.15}$$

this is found to be

$$J[\underline{\xi}] = \text{col.}[\{\frac{\xi_1}{C_1} \int_0^{T_f} \frac{1}{n_1(t)} dt\},\ldots,\{\frac{\xi_{N_h}}{C_{N_h}} \int_0^{T_f} \frac{1}{n_{N_h}(t)} dt\}] \tag{6.4.16}$$

or

$$J[\underline{\xi}] = \underline{\Lambda}\ \underline{\xi} \tag{6.4.17}$$

with

$$\underline{\Lambda} - \text{diag}[(\int_0^{T_f} \frac{1}{C_1 n_1(t)} dt,\ldots,(\int_0^{T_f} \frac{1}{C_{N_h} n_{N_h}(t)} dt)] \tag{6.4.18}$$

so that the operation $\underline{J^{-1}\xi}$ is given by:

$$J^{-1}(\underline{\xi}) = \underline{\Lambda}^{-1}\underline{\xi} \tag{6.4.19}$$

This yeilds the pseudo-inverse operation given by:

$$\underline{T^{\dagger}\xi} = T^{*}[\underline{J^{-1}\xi}] \tag{6.4.20}$$

as

$$\underline{T^{\dagger}\xi}|_{\underline{P}} = \underline{0} \tag{6.4.21}$$

$$\underline{T^{\dagger}\xi}|_{\underline{Q}} = \underline{0} \tag{6.4.22}$$

$$\underline{T^{\dagger}\xi}|_{\underline{E}} = \underline{0} \tag{6.4.23}$$

$$\underline{T^{\dagger}\xi}|_{\underline{W}_i} = \text{col.}\left[\frac{\xi_i}{n_i(t)\int_0^{T_f}\frac{1}{n_i(t)}\,dt}, 0\right]$$

$$i = 1,\ldots,N_h \tag{6.4.24}$$

Consider (6.3.74) and let:

$$\underline{V}(t) = \text{col.}[\underline{V}_{\underline{P}}(t),\underline{V}_{\underline{Q}}(t),\underline{V}_{\underline{E}}(t),\underline{V}_{\underline{W}}(t)] \tag{6.4.25}$$

Then using (6.3.72) and (6.3.57) one obtains

$$\underline{V}_{\underline{P}}^T(t) = \underline{L}_{\underline{P}}^T(t)\underline{B}_{\underline{P}}^{-1}(t) \tag{6.4.26}$$

$$\underline{V}_{\underline{Q}}^T(t) = \underline{L}_{\underline{Q}}^T(t)\underline{B}_{\underline{Q}}^{-1}(t) \tag{6.4.27}$$

$$\underline{V}_{\underline{E}}^T(t) = \underline{L}_{\underline{E}}^T(t)\underline{B}_{\underline{E}}^{-1}(t) \tag{6.4.28}$$

$$\underline{V}_{\underline{W}}^T(t) = \underline{L}_{\underline{W}}^T\underline{B}_{\underline{W}}^{-1}(t) \tag{6.4.29}$$

Furthermore, let

$$\underline{V}_{\underline{P}}(t) = \text{col.}[\underline{V}_{\underline{P}_h}(t),\underline{V}_{\underline{P}_s}(t),V_N(t)] \tag{6.4.30}$$

using (6.3.42) and (6.3.57) then one obtains:

$$\underline{V}_p^T(t) = [(\underline{L}_{P_h}^T(t)\underline{B}_{P_h}^{-1}(t))(\underline{L}_{P_s}^T(t)\underline{B}_{P_s}^{-1}(t))] \qquad (6.4.31)$$

Thus

$$\underline{V}_{P_h}^T(t) = \underline{L}_{P_h}^T(t)\underline{B}_{P_h}^{-1}(t) \qquad (6.4.32)$$

$$\underline{V}_{P_s}^T(t) = \underline{L}_{P_s}^T(t)\underline{B}_{P_s}^{-1}(t) \qquad (6.4.33)$$

From (6.3.50) and (6.3.58) one obtains (6.4.32):

$$\underline{V}_{P_h}(t) = \text{col.}\left[\frac{n_i(t) + \ell_i'(t) - \ell_i(t) - \lambda_{p_i}(t)}{M_i(t)}\right] \qquad i = 1,\ldots,N_h \qquad (6.4.34)$$

and from (6.3.51) and (6.3.60) for (6.4.33):

$$\underline{V}_{P_s}(t) = \text{col.}\left[\frac{\beta_i + \ell_i'(t) - \ell_i(t) - \lambda_{p_i}(t)}{M_i(t) + \gamma_i}\right] \qquad i = N_h+1,\ldots,N_g \qquad (6.4.35)$$

The matrix inverses $\underline{B}_Q^{-1}(t)$ and $\underline{B}_E^{-1}(t)$ will be assumed as:

$$\underline{D}_Q(t) = \underline{B}_Q^{-1}(t) \qquad (6.4.36)$$

$$\underline{D}_E(t) = B_E^{-1}(t) \qquad (6.4.37)$$

with

$$\underline{D}_Q(t) = (d_{Q_{ij}}(t)) \qquad (6.4.38)$$

$$\underline{D}_E(t) = (d_{E_{ij}}(t)) \qquad (6.4.39)$$

Note that the $d_{Q_{ij}}(t)$'s are functions of $M_i(t)$ and the $d_{E_{ij}}(t)$'s are functions of $\lambda_{p_i}(t)$ and $\lambda_{q_i}(t)$. Thus $\underline{V}_Q(t)$ and $\underline{V}_E(t)$ are given by:

$$\underline{V}_Q(t) = \text{col.}[V_{Q_1}(t),\ldots,V_{Q_{N_g}}(t)] \tag{6.4.40}$$

and

$$\underline{V}_E(t) = \text{col.}[V_{E_{d_1}}(t),\ldots,V_{E_{d_N}},V_{E_{q_1}}(t),\ldots,V_{E_{q_N}}(t)] \tag{6.4.41}$$

Note $V_{E_{d_{N_g}}}$ and $V_{E_{q_{N_g}}}$ are not present in $\underline{V}_E(t)$.

Then (6.3.52), (6.4.24) and (6.4.38) yield:

$$V_{Q_i}(t) = \sum_{j=1}^{N_g}[e_j'(t) - e_j(t) + \lambda_{q_j}(t)]d_{Q_{ij}}(t)$$

$$i = 1,\ldots,N_g \tag{6.4.42}$$

Also (6.3.53), (6.3.54), (6.4.28), (6.4.41) and (6.4.42) yield

$$V_{E_{d_i}}(t) = \sum_{\substack{j=1\\ j\neq N_g}}^{N-1} [a_{N_g,j}(t) + a_{j,N_g}(t)]E_{d_{N_g}}(t)d_{E_{ij}}(t)$$

$$+ \sum_{\substack{j=N\\ j\neq N+N_g}}^{2(N-1)} -[b_{N_g,j-N_g+1}(t) + b_{j-N+1,N_g}(t)]$$

$$E_{d_{N_g}}(t)d_{E_{ij}}(t) \qquad i = 1,\ldots,N \tag{6.4.43}$$

$$V_{E_{q_i}}(t) = \sum_{\substack{j=1\\ j\neq N_g}}^{N-1} [a_{N_g,j}(t) + a_{j,N_g}(t)]E_{d_{N_g}} d_{E_{(N_g-1+i)j}}(t)$$

$$+ \sum_{\substack{j=N\\ j\neq N+N_g}}^{2(N-1)} -[b_{N_g,j-N_g+1}(t) + b_{j-N_g+1,N_g}(t)]E_{d_{N_g}}(t)$$

$$d_{E_{(N_g-1+i)j}}(t) \qquad i = 1,\ldots,N \tag{6.4.44}$$

Consider (6.4.29) and let

$$\underline{V}_W(t) = \text{col.}[\underline{V}_{W_1}(t),\ldots,\underline{V}_{W_{N_h}}(t)] \tag{6.4.45}$$

then using (6.3.73) one obtains

$$\underline{V}^T_{W_i}(t) = \underline{L}^T_{W_i}(t)\underline{B}^{-1}_{W_i}(t) \qquad i = 1,\ldots,N_h \tag{6.4.46}$$

if $\underline{V}_{W_i}(t) = \text{col.}[V_{W_{q_i}}(t), V_{W_{Q_i}}(t)]$

then (6.3.55) and (6.3.65) yield:

$$V_{W_{q_i}}(t) = \frac{m_i(t) + n_i(t)A_i(t)}{C_i n_i(t)} \qquad i = 1,\ldots,N_h \tag{6.4.47}$$

$$V_{W_{Q_i}}(t) = -\frac{2\dot{m}_i(t)}{B_i\dot{n}_i(t)} \qquad i = 1,\ldots,N_h \tag{6.4.48}$$

The optimal solution as given in (6.4.1) is written in component form

$$\underline{P}_\xi(t) = T^\dagger[\underline{b} + T(\frac{\underline{V}(t)}{2})]|_{\underline{P}} - \frac{\underline{V}_P(t)}{2} \tag{6.4.49}$$

$$\underline{Q}_\xi(t) = T^\dagger[\underline{b} + T(\frac{\underline{V}(t)}{2})]|_{\underline{Q}} - \frac{\underline{V}_Q(t)}{2} \tag{6.4.50}$$

$$\underline{E}_\xi(t) = T^\dagger[\underline{b} + T(\frac{\underline{V}(t)}{2})]|_{\underline{E}} - \frac{\underline{V}_E(t)}{2} \tag{6.4.51}$$

$$\underline{W}_\xi(t) = T^\dagger[\underline{b} + T(\frac{\underline{V}(t)}{2})]|_{\underline{W}} - \frac{\underline{V}_W(t)}{2} \tag{6.4.52}$$

From (6.4.21) through (6.4.23) one obtains for (6.4.49) through (6.4.52)

$$\underline{P}_\xi(t) = -\frac{\underline{V}_P(t)}{2} \tag{6.4.53}$$

$$\underline{Q}_\xi(t) = -\frac{\underline{V}_Q(t)}{2} \tag{6.4.54}$$

$$\underline{E}_\xi(t) = -\frac{\underline{V}_E(t)}{2} \tag{6.4.55}$$

Moreover $\underline{W}_{\xi}(t)$ in (6.4.52) is rewritten component-wise as:

$$\underline{W}_{\xi_i}(t) = T^{\dagger}[\underline{b} + T(\frac{\underline{V}(t)}{2})]|_{\underline{W}_i} - \frac{V_{W_i}(t)}{2} \qquad (6.4.56)$$

using (6.4.24), (6.4.49) and (6.4.50) this yields:

$$q_{\xi_i}(t) = -\frac{[m_i(t) + n_i(t)A_i(t)]}{2C_{W_i}n_i(t)} + \frac{b_i + \int_0^{T_f} \frac{m_i(t) + n_i(t)A_i(t)}{2C_i n_i(t)}\,dt}{n_i(t)\int_0^{T_f} \frac{1}{n_i(t)}\,dt}$$

$$i = 1,\ldots,N_h \qquad (6.4.57)$$

$$Q_{W_{\xi_i}}(t) = \frac{\dot{m}_i(t)}{B_{W_i}\dot{n}_i(t)} \qquad i = 1,\ldots,N_h \qquad (6.4.58)$$

Furthermore, using (6.4.33), (6.4.34) and (6.4.35) in (6.4.53) one obtains:

$$P_{h_{i_\xi}}(t) = -\frac{[n_i(t) + \ell_i'(t) - \ell_i(t) - \lambda_{p_i}(t)]}{2M_i(t)}$$

$$i = 1,\ldots,N_h \qquad (6.4.59)$$

$$P_{S_{i_\xi}}(t) = -\frac{[\beta_i + \ell_i'(t) - \ell_i(t) - \lambda_{p_i}(t)]}{2[M_i(t) + \gamma_i]}$$

$$i = N_h+1,\ldots,N_G$$

$$(6.4.60)$$

Also from (6.4.42)

$$Q_{i_\xi}(t) = -\frac{1}{2}\left[\sum_{j=1}^{N_g} [e_j'(t) - e_j(t) + \lambda_{q_j}(t)]d_{Q_{ij}}(t)\right.$$

$$i = 1,\ldots,N_g \qquad (6.4.61)$$

Finally, using (6.4.43) and (6.4.44) the following is obtained

$$E_{d_{i_\xi}}(t) = -\frac{E_{d_{N_g}}(t)}{2}\left[\sum_{\substack{j=1 \\ j\neq N_g}}^{N} [a_{N_g,j}(t) + a_{j,N_g}(t)]d_{E_{ij}}(t)\right.$$

$$\left.+\sum_{\substack{j=N \\ j\neq N_g+N}}^{2(N-1)} [b_{N_g,j-N_g+1}(t) + b_{j-N_g+1,N_g}(t)]d_{E_{ij}}(t)\right]$$

$$i = 1,\ldots,N \qquad (6.4.62)$$

$$E_{q_{i_\xi}}(t) = -\frac{E_{d_{N_g}}(t)}{2}\left[\sum_{\substack{j=1 \\ j\neq N_g}}^{N} [a_{N_g,j}(t) + a_{j,N_g}(t)]d_{E_{(N_g-1+i),j}}(t)\right.$$

$$\left.-\sum_{\substack{j=N \\ j\neq N+N_g}}^{2(N)} [b_{n_g,j-N_g+1}(t) + b_{j-N_g+1,N_g}(t)]d_{E_{(N_g-1+i),j}}(t)\right]$$

$$i = 1,\ldots,N \qquad (6.4.63)$$

The last two equations are modified by noting that the symmetric matrix $\underline{B}_E$ in (6.3.69) can be written as:

$$\underline{B}_E(t) = \begin{bmatrix} \underline{A}_E(t) & \underline{C}_E(t) \\ \underline{C}_E^T(t) & \underline{A}_E(t) \end{bmatrix} \qquad (6.4.64)$$

$$\underline{A}_E(t) = (a_{ij_E}(t))_{(N-1)x(N-1)} \tag{6.4.65}$$

$$\underline{C}_E(t) = (C_{ij_E}(t))_{(N-1)x(N-1)} \tag{6.4.66}$$

$$a_{ij_E}(t) = \frac{a_{ij}(t) + a_{ji}(t)}{2} \tag{6.4.67}$$

$$C_{ij_E}(t) = \frac{-b_{ij}(t) + b_{ji}(t)}{2} \tag{6.4.68}$$

Note that $\underline{A}_E(t)$ is symmetric but $\underline{C}_E(t)$ is not. Moreover $\underline{C}_E(t) = - C_E^T(t)$.

The inverse matrix $B_E^{-1}(t)$ is denoted by

$$\underline{D}_E(t) = B_E^{-1}(t) \tag{6.4.69}$$

Then using (6.4.67) and the property $\underline{C}_E(t) = - C_E^T(t)$, one obtains for $\underline{D}_E(t)$:

$$\underline{D}_E(t) = \begin{bmatrix} F_E(t) & H_E(t) \\ -\underline{H}_E(t) & \underline{F}_E(t) \end{bmatrix} \tag{6.4.70}$$

where

$$\underline{F}_E(t) = [\underline{A}_E(t) + \underline{C}_E(t)\underline{A}_E^{-1}(t)\underline{C}_E(t)]^{-1} \tag{6.4.71}$$

$$\underline{H}_E(t) = -\underline{A}_E^{-1}(t)\underline{C}_E(t)\underline{F}_E(t) \tag{6.4.72}$$

Thus

$$\underline{V}_{E_d}(t) = \underline{L}_{E_d}(t)\underline{F}_E(t) - \underline{L}_{E_q}(t)\underline{H}_E(t) \tag{6.4.73}$$

$$\underline{V}_{E_q}(t) = \underline{L}_{E_d}(t)\underline{H}_E(t) + \underline{L}_{E_q}(t)\underline{F}_E(t) \tag{6.4.74}$$

so that (6.4.65) and (6.4.66) can be written in the alternative form:

$$E_{d_{i_\xi}}(t) = -\frac{E_{d_{N_g}}(t)}{2}\Big[\sum_{\substack{j=1\\ j\neq N_g}}^{N}(a_{N_g,j}(t) + a_{j,N_g}(t))f_{ij}(t)$$

$$+\sum_{\substack{j=1\\ j\neq N_g}}^{N}(b_{N_g,j}(t) + b_{j,N_g}(t))h_{ij}(t)\Big] \tag{6.4.75}$$

$$E_{q_{i_\xi}}(t) = -\frac{E_{d_{N_g}}(t)}{2}\Big[\sum_{\substack{j=1\\ j\neq N_g}}^{N}(a_{N_g,j}(t) + a_{j,N_g}(t))h_{ij}(t)$$

$$-\sum_{\substack{j=1\\ j\neq N_g}}^{N}(b_{N_g,j}(t) + b_{j,N_g}(t))f_{ij}(t)\Big] \tag{6.4.76}$$

6.5 Implementing the Optimal Solution

The method suggested for actually implementing the optimal solution is best illustrated by way of an example. The example concerns a practical power system as shown in Fig. (6.1). Here bus number one is a hydro-generator bus, bus number two is a thermal plant's bus (also slack bus) and bus number three is a load bus. The classification of the various variables in the sample system is given in Table (6.1).

The reliability matrix $\underline{K}$ cost functional (6.2.16) is taken as

$$\underline{K} = \begin{bmatrix} \frac{1+K_1}{2} & -\frac{K_1}{2} \\ -\frac{K_1}{2} & \frac{1+K_1}{2} \end{bmatrix} \tag{6.5.1}$$

$$K_1 = 0.02 \tag{6.5.2}$$

This is the same reliability matrix as the one given in [64]. Thus the

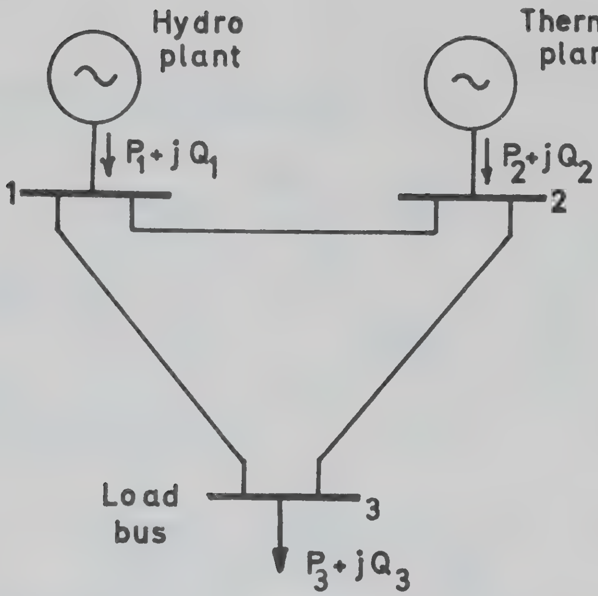


Figure 6.1 EXAMPLE 3-BUS SYSTEM

Bus	Type	E	E_d	E_q	P	Q	q	$\int_0^{T_f} q(t)dt$
1	Hydro	Optimum	Optimum	Optimum	Optimum	Optimum	Optimum	Specified
2	Thermal (Slack)	Specified	Specified	Specified	Optimum	Optimum		
3	Load	Optimum	Optimum	Optimum	Specified	Specified		

TABLE 6.1 CLASSIFICATION OF VARIABLES IN THE EXAMPLE SYSTEM.

matrix $\underline{B}_Q(t)$ in (6.3.61) is given by

$$\underline{B}_Q(t) = \begin{bmatrix} (M_1(t) + \frac{1+K_1}{2}) & (-\frac{K_1}{2}) \\ (-\frac{K_1}{2}) & (M_2(t) + \frac{1+K_1}{2}) \end{bmatrix} \tag{6.5.3}$$

and the inverse matrix $\underline{D}_Q(t)$ is:

$$\underline{D}_Q(t) = \begin{bmatrix} \frac{M_2(t) + (\frac{1+K_1}{2})}{\Delta_Q} & \frac{K_1}{2\Delta_Q} \\ \frac{K_1}{2\Delta_Q} & \frac{M_1(t) + (\frac{1+K_1}{2})}{\Delta_Q} \end{bmatrix} \tag{6.5.4}$$

$$\Delta_Q = M_1(t)M_2(t) + \frac{1+K_1}{2}(M_1(t) + M_2(t)) + \frac{1+2K_1}{4} \tag{6.5.5}$$

The optimum active and reactive powers at buses 1 and 2 are given by (6.4.59) through (6.4.61) as:

$$P_{1_\xi}(t) = -\frac{[n_1(t) + \ell_1'(t) - \ell_1(t) - \lambda_{p_1}(t)}{2M_1(t)} \tag{6.5.6}$$

$$P_{2_\xi}(t) = -\frac{[\beta_2 + \ell_2'(t) - \ell_2(t) - \lambda_{p_2}(t)]}{2[M_2(t) + \gamma_2]} \tag{6.5.7}$$

$$Q_{1_\xi}(t) = -\frac{1}{2\Delta_Q}[(e_1'(t) - e_1(t) + \lambda_{q_1}(t))(M_2(t) + (\frac{1+K_1}{2}))$$

$$+ (e_2'(t) - e_2(t) + \lambda_{q_2}(t))\,\frac{K_1}{2}] \tag{6.5.8}$$

$$Q_{2_\xi}(t) = -\frac{1}{2\Delta_Q}[(e_1'(t) - e_1(t) + \lambda_{q_1}(t))\frac{K_1}{2}$$

$$+ (e_2'(t) - e_2(t) + \lambda_{q_2}(t))(M_1(t) + (\frac{1+K_1}{2}))] \tag{6.5.9}$$

According to (6.3.66), the matrix $\underline{B}_{E_o}(t)$ is given by

$$\underline{B}_{E_o}(t) = \left[\begin{array}{cc|cc} a_{11}(t) & a_{13}(t) & 0 & -b_{13}(t) \\ a_{31}(t) & a_{33}(t) & -b_{31}(t) & 0 \\ \hline 0 & b_{13}(t) & a_{11}(t) & a_{13}(t) \\ b_{31}(t) & 0 & a_{31}(t) & a_{33}(t) \end{array}\right] \tag{6.5.10}$$

And the symmetric matrix $B_E(t)$ is then

$$\underline{B}_E(t) = \left[\begin{array}{cc|cc} a_{11}(t) & [\frac{a_{13}(t) + a_{31}(t)}{2}] & 0 & \frac{b_{31}(t) - b_{13}(t)}{2} \\ \frac{a_{13}(t) + a_{31}(t)}{2} & a_{33}(t) & \frac{b_{13}(t) - b_{31}(t)}{2} & 0 \\ \hline 0 & \frac{b_{13}(t) - b_{31}(t)}{2} & a_{11}(t) & \frac{a_{13}(t) + a_{31}(t)}{2} \\ \frac{b_{31}(t) - b_{13}(t)}{2} & 0 & \frac{a_{13}(t) + a_{31}(t)}{2} & a_{33}(t) \end{array}\right] \tag{6.5.11}$$

Thus in accordance with (6.4.64) one obtains

$$\underline{A}_E(t) = \begin{bmatrix} a_{11}(t) & \frac{a_{13}(t) + a_{31}(t)}{2} \\ \frac{a_{13}(t) + a_{31}(t)}{2} & a_{33}(t) \end{bmatrix} \tag{6.5.12}$$

$$\underline{C}_E(t) = \begin{bmatrix} 0 & \frac{b_{31}(t) - b_{13}(t)}{2} \\ \frac{b_{13}(t) - b_{31}(t)}{2} & 0 \end{bmatrix} \tag{6.5.13}$$

Thus from (6.4.12) and (6.4.13) in (6.4.71) one obtains:

$$F_E(t) = \frac{1}{\Delta_E'(t)} \begin{bmatrix} a_{33}(t) & -[\frac{a_{13}(t) + a_{31}(t)}{2}] \\ -[\frac{a_{13}(t) + a_{31}(t)}{2}] & a_{11}(t) \end{bmatrix} \tag{6.5.14}$$

$$\Delta_E' = a_{11}(t)a_{33}(t) - (\frac{a_{13}(t) + a_{31}(t)}{2})^2 - [\frac{b_{31}(t) - b_{13}(t)}{2}]^2 \tag{6.5.15}$$

Also using (6.4.72):

$$\underline{H}_E(t) = \frac{1}{\Delta_E'(t)} \begin{bmatrix} 0 & \frac{b_{13}(t) - b_{31}(t)}{2} \\ \frac{b_{31}(t) - b_{13}(t)}{2} & 0 \end{bmatrix} \tag{6.5.16}$$

Using (6.5.14) through (6.5.16) in (6.4.75) and (6.4.76) one obtains:

$$E_{d_{1_\xi}}(t) = -\frac{E_{d_2}(t)}{2\Delta_E'(t)} [(a_{21}(t) + a_{12}(t))a_{33}(t) - (a_{23}(t) + a_{32}(t))(\frac{a_{13}(t) - a_{31}(t)}{2})$$

$$+ (b_{23}(t) + b_{32}(t))(\frac{b_{31}(t) - b_{13}(t)}{2})] \tag{6.5.17}$$

$$E_{d_{3_\xi}} = - \frac{E_{d_2}(t)}{2\Delta_E'} [-(a_{21}(t) + a_{12}(t)(\frac{a_{13}(t) + a_{31}(t)}{2})$$

$$+ (a_{23}(t) + a_{32}(t))a_{11}(t)$$

$$- (b_{21}(t) + b_{12}(t))(\frac{b_{31}(t) - b_{13}(t)}{2})] \tag{6.5.18}$$

$$E_{q_{1_\xi}}(t) = - \frac{E_{d_2}(t)}{2\Delta_E'} [(a_{23}(t) + a_{32}(t))(\frac{b_{31}(t) - b_{13}(t)}{2})$$

$$- (b_{21}(t) + b_{12}(t))a_{33}(t) + (b_{23}(t) + b_{32}(t))$$

$$(\frac{a_{13}(t) + a_{31}(t)}{2})] \tag{6.5.19}$$

$$E_{q_{3_\xi}}(t) = - \frac{E_{d_2}(t)}{2\Delta_E'} [-(a_{21}(t) + a_{12}(t))(\frac{b_{31}(t) - b_{13}(t)}{2})$$

$$+ (b_{21}(t) + b_{12}(t))(\frac{a_{13}(t) + a_{31}(t)}{2})$$

$$- (b_{23}(t) + b_{32}(t))a_{11}(t)] \tag{6.5.20}$$

The optimal expressions for $q_1(t)$ and $Q_{W_1}(t)$ as given by (6.4.57) and (6.4.58) can be combined to yield:

$$\frac{d}{dt}[2C_{W_1} n_1(t)\dot{Q}_{W_1}(t) + n_1(t)A_1(t)] + B_{W_1}\dot{n}_1(t)Q_{W_1}(t) = 0$$

$$Q_{W_1}(o) = 0 \qquad Q_{W_1}(T_f) = b_1 \tag{6.5.21}$$

Here the optimal values for the unknown variables are obtained in terms of the functions $\lambda_{p_1}(t)$, $\lambda_{p_2}(t)$, $\lambda_{p_3}(t)$, $\lambda_{q_1}(t)$, $\lambda_{q_2}(t)$, $\lambda_{q_3}(t)$ and $n_1(t)$. These are to be determined such that the equality constraints (6.2.12) for i = 1,2,3, (6.2.13) for i = 1,2,3 and (6.2.22) for i = 1 are satisfied. Note that the Kuhn-Tucker multipliers $M_i(t)$, $\ell_i(t)$, $\ell_i'(t)$, $e_i(t)$ and $e_i'(t)$ for i = 1,2, are determined in accordance with the exclusion equations (6.3.12) through (6.3.16).

In the case when none of the inequality constraints is violated, the Kuhn-Tucker multipliers are zeros. Thus (6.5.7) yields

$$P_{2_\xi}(t) = \frac{\lambda_{p_2}(t) - \beta_2}{2\gamma_2} \tag{6.5.22}$$

(6.5.8) and (6.5.9) reduce to:

$$Q_{1_\xi}(t) = -\frac{1}{1+2K_1}[(1+K_1)\lambda_{q_1}(t) + K_1\lambda_{q_2}(t)] \tag{6.5.23}$$

$$Q_{2_\xi}(t) = -\frac{1}{1+2K_1}[K_1\lambda_{q_1}(t) + (1+K_1)\lambda_{q_2}(t)] \tag{6.5.24}$$

while (6.5.6) yields

$$n_1(t) = \lambda_{p_1}(t) \tag{6.5.25}$$

Note that this result can be obtained if $P_1(t)$ in the formulation of the problem was not taken as a control. This is the case when there is no second order term in $P_1(t)$ in the cost functional or the associated constraints.

The number of unknown variables in this example is 16, these are divided into two categories:

1. <u>Physical variables</u>: these are $P_{1_\xi}(t)$, $P_{2_\xi}(t)$, $Q_{1_\xi}(t)$, $Q_{2_\xi}(t)$, $E_{d_{1_\xi}}(t)$, $E_{q_{1_\xi}}(t)$, $E_{d_{3_\xi}}(t)$ and $Q_{W_{1_\xi}}(t)$. The physical variables are related directly by the load flow equations (6.2.12) and (6.2.13) for $i = 1,2,3$ and the hydro power equation (6.2.22). Note here that the number of the physical variables is 9 while the constraining equations are 7.

2. <u>Dual variables</u>: these are $\lambda_{p_i}(t)$, $\lambda_{q_i}(t)$ $(i = 1,2,3)$ and $n_i(t)$. These are related to the physical variables by the optimal expressions (6.5.6) through (6.5.9), (6.5.17) through (6.5.20) and (6.5.21). The number of the dual variables is 7 and the constraining equations are 9. Thus one has 16 equations in 16 unknowns.

It is possible in this example to reduce the number of equations and unknown variables by algebraic manipulations. However, this will not be done for many reasons. One of the reasons is that the reduced equations for this example are not easily extended to higher dimensions. Another reason is that it is important to keep the variables as they are in order to satisfy the inequality constraints and the corresponding exclusion equations. The third and most important one is that the load flow equations can be solved easily with the present state of the art. Thus developing a computer program that is an extension of a load flow program is highly desirable.

The Example 3-bus System's particulars are summarized in Table 6.2. The results of the optimum load flow study are shown in Fig. (6.2) through (6.5).

TABLE 6.2

EXAMPLE 3-NODE SYSTEM'S PARTICULARS

The Thermal Plant Cost:

$\beta = 4.0$ $\gamma = 1.2\text{x}10^{-3}$

The Electric Network: (Admittances in mhos).

$G^{ij} = 0.147\text{x}10^{-2}$ $B^{ij} = -0.63\text{x}10^{-2}$ $i,j - 1,2,3$ $i \neq j$

$G^{io} = 0.0$ $B^{io} = 0.05\text{x}10^{-2}$ $i = 1,2,3$

The Electric Variables Specifications: (Voltages line to line).

Slack Bus:	$E_{d_2}(t) = 220$ KV	$E_{q_2}(t) = 0.0$
Load Node:	$P_3(t) = -41.5$MW	$Q_3(t) = -14$ MVAR
Load at Node 1:	$P_{D_1}(t) = -50$ MW	$Q_{D_1}(t) = +30$ MVAR
Load at Node 2:	$P_{D_2}(t) = -50$ MW	$Q_{D_2}(t) = -75$ MVAR

The Hydro Plant:

$\eta = 0.708$ $V = 13.35\text{x}10^8 \text{ft}^3$

$\beta_y = 1/7.2\text{x}10^{10}$ $\beta_T = 0.0$

$S(o) = 7.2\text{x}10^{12}$

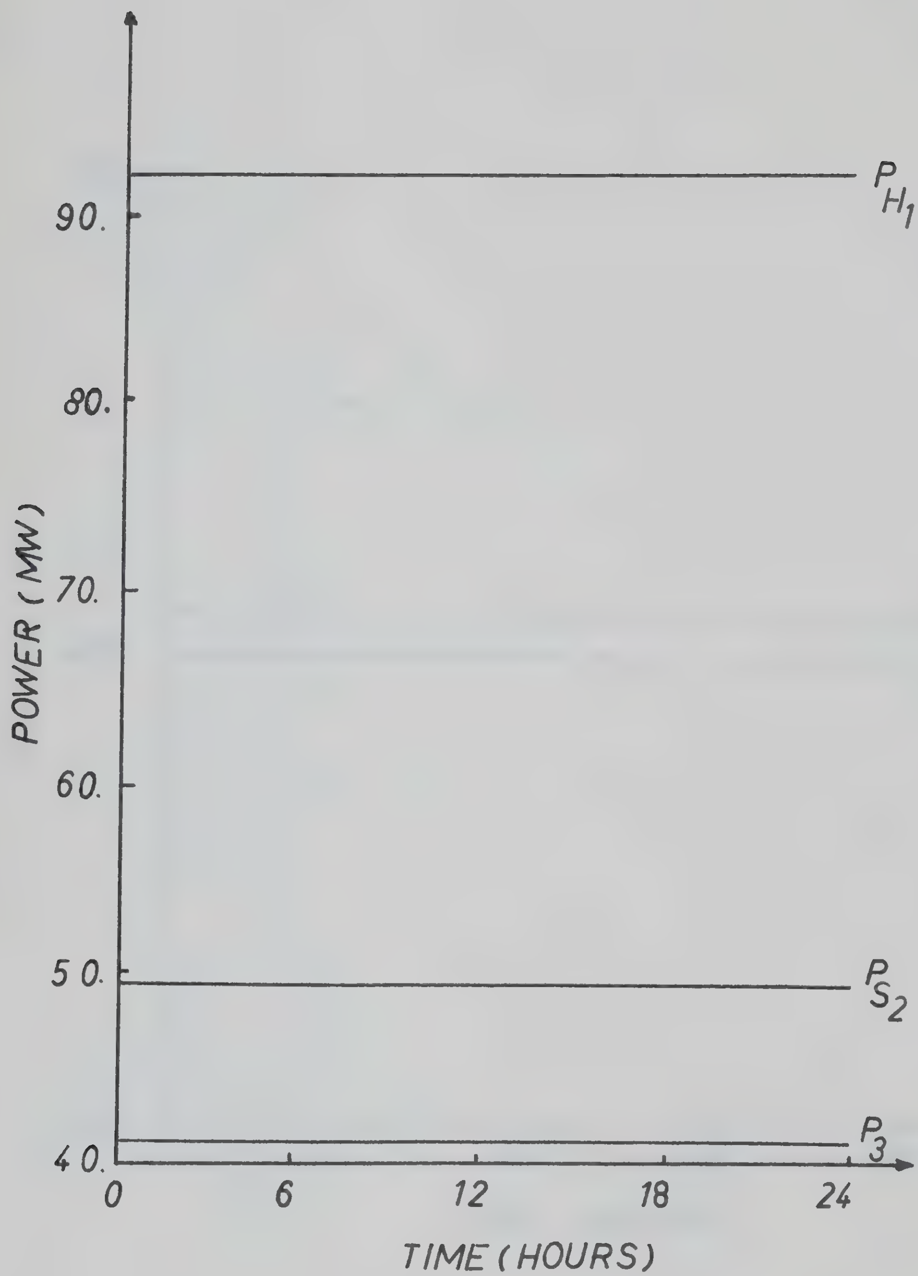


Figure 6.2 ACTIVE Power Variations.

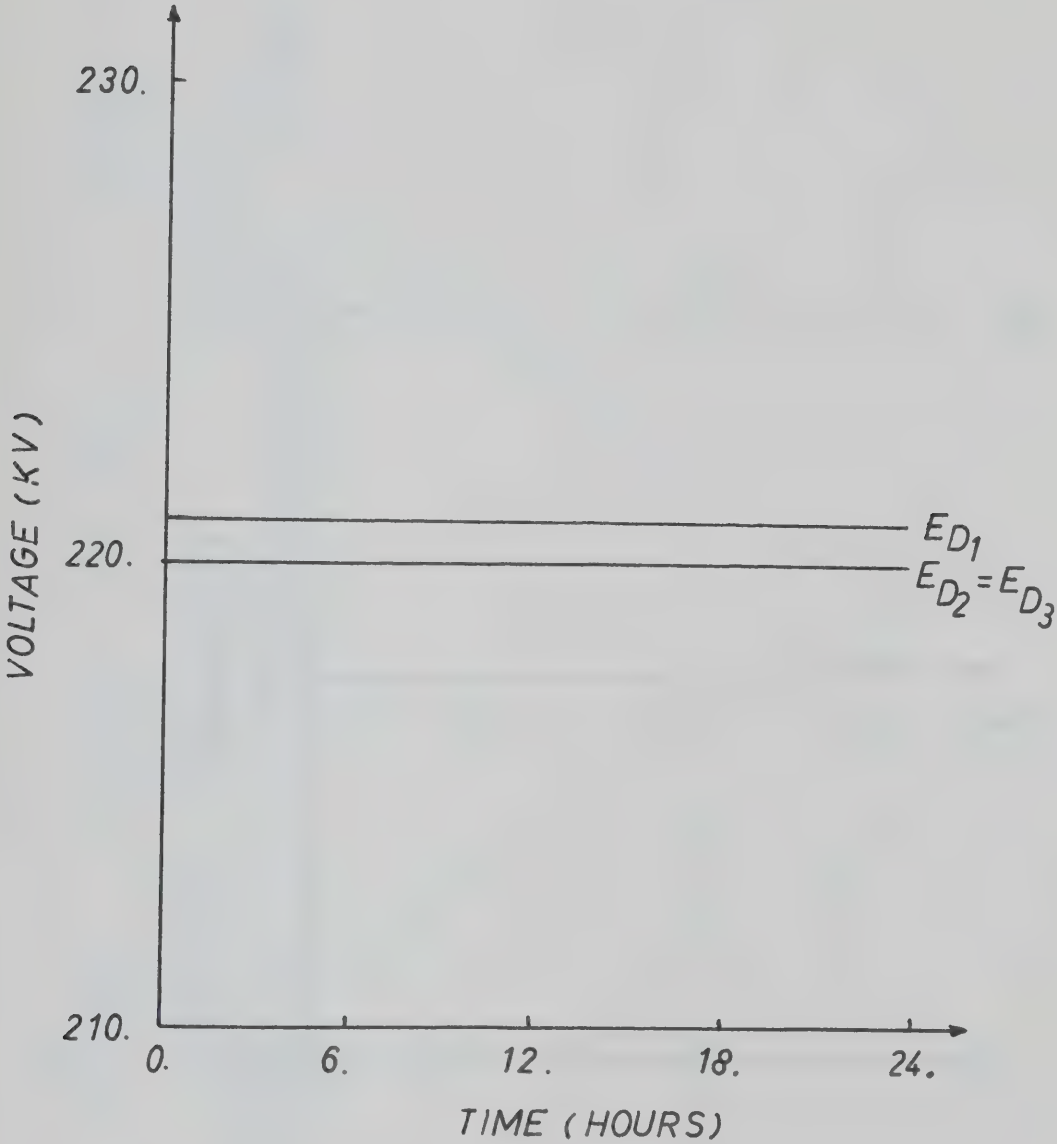


Figure 6.3 Variation of the Direct Components of the Voltages.

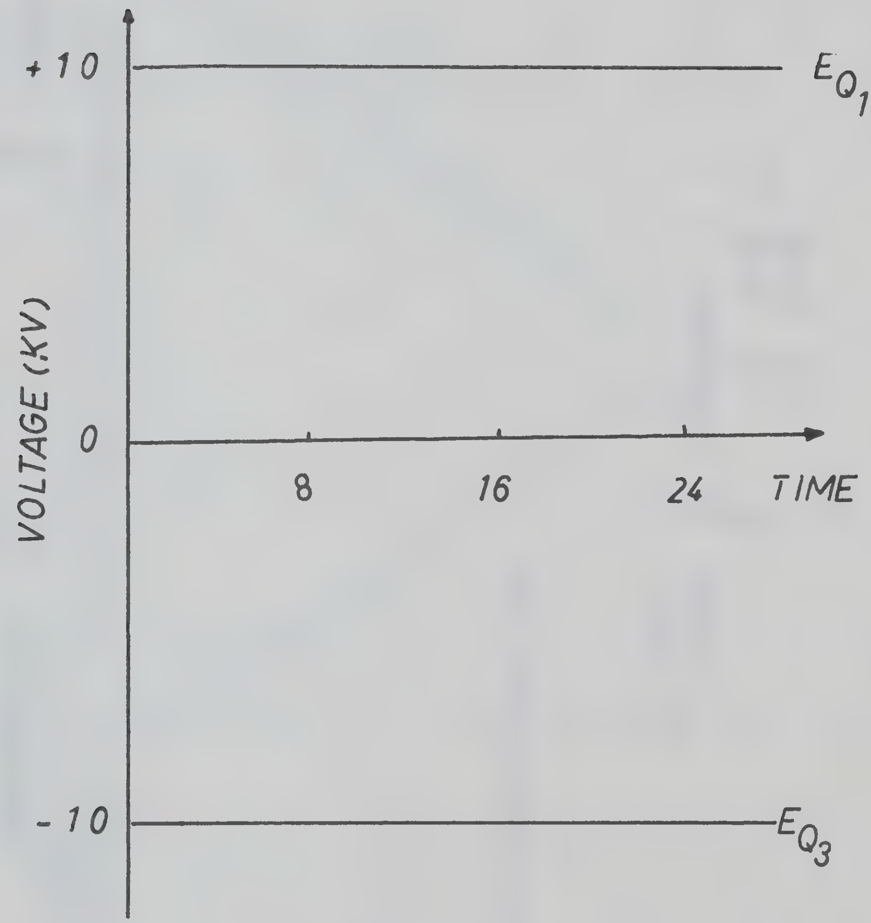
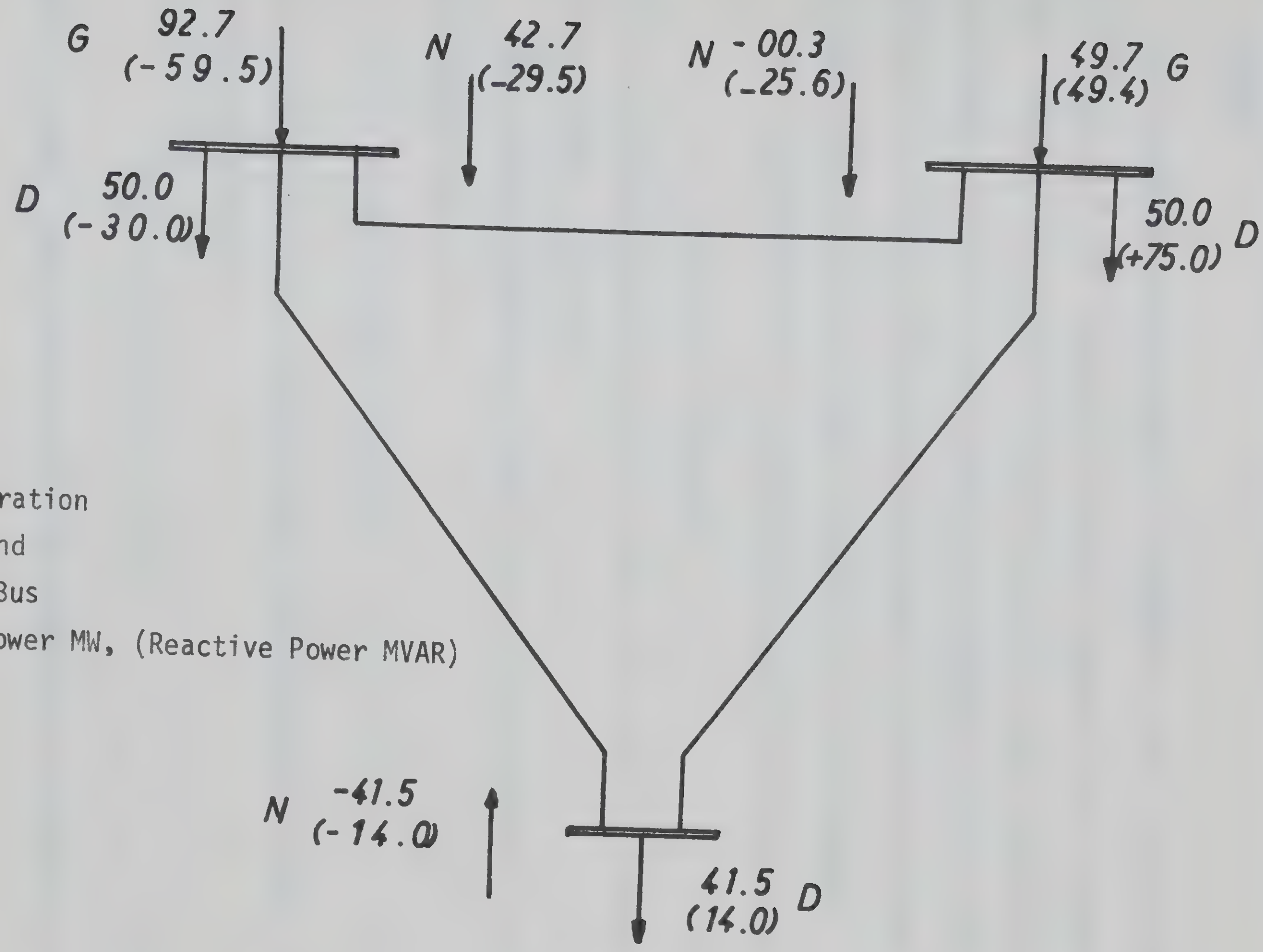


Figure 6.4 Variation of Quadrature Components of the Voltages.



G = Generation

D = Demand

N = Net Bus

Active Power MW, (Reactive Power MVAR)

Figure 6.5 Active and Reactive Powers in the Sample System.

6.6 Trapezoidal Reservoirs and Variable Efficiency Hydro-plants Considerations

In formulating the problems in Chapters 5 and 6 it was assumed that the efficiency of each hydro-plant remains constant over the operating range. Another assumption is that of vertical sided reservoirs at the hydro-plants. In this section these two assumptions are relaxed and the modifications to both formulation and optimal solution are shown. Here the power system problem stated in Section 6.2 is considered.

The ith hydro-plant's active power generation is given by:

$$P_{h_i}(t)G_i(t) = h_i(t)q_i(t) \tag{6.6.1}$$

This is precisely equation (5.2.13) except that here the inverse efficiency G_i is no longer a constant.

The effective hydraulic head at the ith hydro-plant is given by (5.2.2) which is:

$$h_i(t) = y_i(t) - y_{T_i}(t) \tag{6.6.2}$$

The forebay elevation is related to the forebay volume of water stored $S_i(t)$ by the relation:

$$S_i(t) = \alpha_{y_i} y_i^2(t) + \beta_{y_i} y_i(t) \tag{6.6.3}$$

where α_{y_i} and β_{y_i} are constants for the trapezoidal reservoir given by:

$$\alpha_{y_i} = \ell_i \tan\phi_i$$

$$\beta_{y_i} = \ell_i b_{o_i}$$

The geometry of a trapezoidal reservoir is shown in Fig.(6.5). The volume of water stored $S_i(t)$ is also given by:

$$S_i(t) = S_i(o) + \int_0^t i_i(\sigma)d\sigma - \int_0^t q_i(\sigma)d\sigma \qquad (6.6.4)$$

This is the reservoir's dynamic equation in the case of no hydraulic coupling between the plants. Here $i_i(t)$ and $q_i(t)$ are the rates of water inflow and discharge respectively. The volume of water stored variable $S_i(t)$ can be eliminated by substituting (6.6.3) in (6.6.4) to obtain:

$$\alpha_{y_i} y_i^2(t) + \beta_{y_i} y_i(t) + \int_0^t q_i(\sigma)d\sigma - S_i(o) - \int_0^t i_i(\sigma)d\sigma = 0 \qquad (6.6.5)$$

The tail-race elevation $y_{T_i}(t)$ is given by:

$$y_{T_i}(t) = y_{T_{io}} + \beta_{T_i} q_i(t) \qquad (6.6.6)$$

This is the same equation as (5.2.4)

Thus the active power generation equation (6.6.1) becomes:

$$P_{h_i}(t)G_i(t) + y_{T_{io}} q_i(t) + \beta_{T_i} q_i^2(t) - q_i(t)y_i(t) = 0 \qquad (6.6.7)$$

Here (6.6.2) and (6.6.6) were substituted in (6.6.1) to obtain (6.6.7).

For all practical purposes the variation of the inverse efficiency $G_i(t)$ with the active power generation can be represented by

$$\alpha_{g_i} G_i^2(t) + \beta_{g_i} G_i(t) + \gamma_{g_i} P_{h_i}^2(t) + \delta_{g_i} P_{h_i}(t) + \theta_{g_i} = 0 \qquad (6.6.8)$$

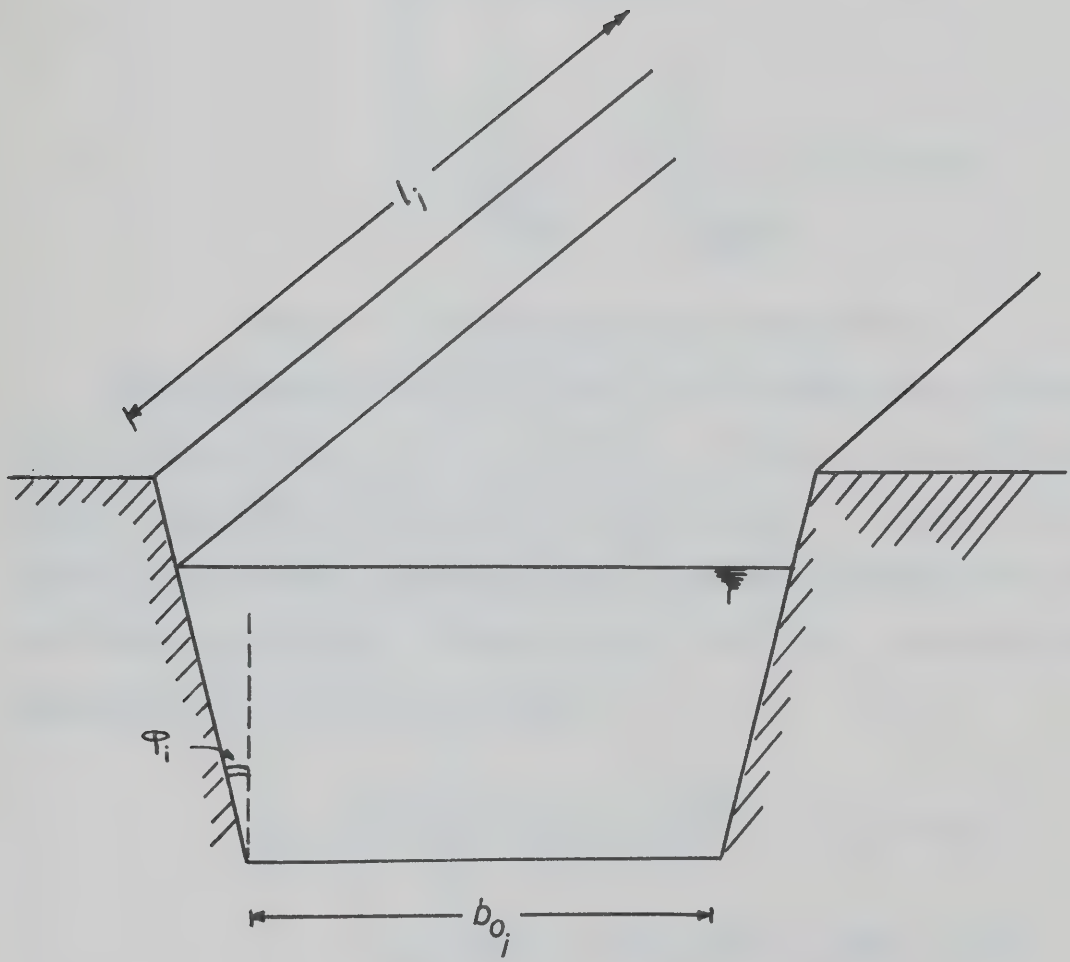


Figure 6.6 A Trapezoidal Reservoir.

This is the equation of an ellipse and is assumed to hold true over the operating range ($P_i^m \leq P_{h_i}(t) \leq P_i^M$) of the hydro-plant. This is shown in Fig.(6.7).

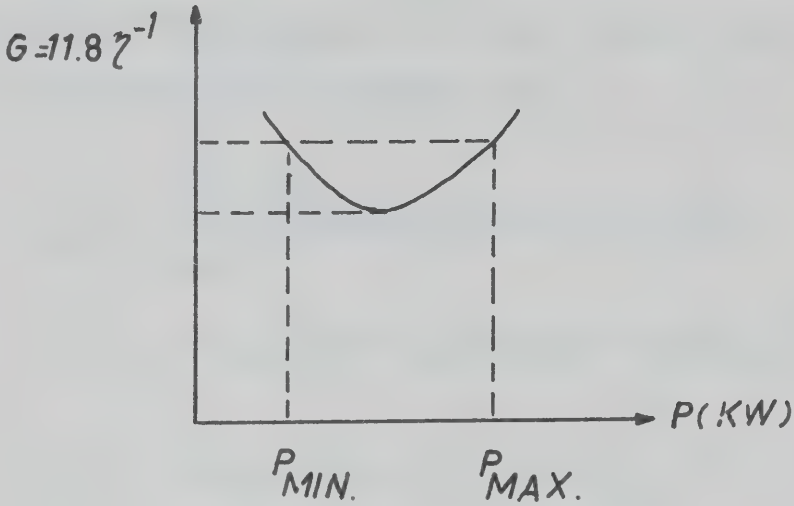


Figure 6.7 Efficiency versus Power Output.

The performance of the <u>ith</u> hydro-plant is completely specified by the relations (6.6.5), (6.6.7) and (6.6.8). Thus the only modification to the formulation of the problem in Section (6.2) is replacing (6.2.22) by the three relations mentioned above. Note here that two more variables per plant are introduced. These are $y_i(t)$ and $G_i(t)$. Accordingly the functional $J_{o_4}(.)$ in (6.3.5) becomes:

$$
\begin{aligned}
J_{o_4}(.) = \int_0^{T_f} [\sum_{i=1}^{N_h} \{ & n_i(t)[P_{h_i}(t)G_i(t) + y_{T_{io}}(t) q_i(t) \\
& + \beta_{T_i} q_i^2(t) - q_i(t)y_i(t)] + m_i(t)[\alpha_{g_i} G_i^2(t) \\
& + \beta_{g_i} G_i(t) + \gamma_{g_i} P_{h_i}^2(t) + \delta_{g_i} P_{h_i}(t) + \theta_{g_i}] \\
& + r_i(t)[\alpha_{y_i} y_i^2(t) + \beta_{y_i} y_i(t) + \int_0^t q_i(s) d\sigma \\
& - S_i(o) - \int_0^t i_i(\sigma) d\sigma]\}]dt \qquad (6.6.9)
\end{aligned}
$$

Note that the variable $Q_{W_i}(t)$ in (6.3.5) is no longer of use. Thus in effect only one extra variable is needed.

Here the control variables are $P_{h_i}(t)$, $G_i(t)$, $q_i(t)$ and $y_i(t)$. Thus the terms explicitly independent of these variables can be dropped from $J_{o_4}(.)$ so that one needs to consider only

$$
\begin{aligned}
J_{o_4'}(.) = \int_0^{T_f} [\sum_{i=1}^{N_h} \{ & n_i(t)[P_{h_i}(t)G_i(t) + y_{T_{io}} q_i(t) \\
& + \beta_{T_i} q_i^2(t) - q_i(t)y_i(t)] + m_i(t)[\alpha_{g_i} G_i^2(t) \\
& + \beta_{g_i} G_i(t) + \gamma_{g_i} P_{h_i}^2(t) + \delta_{g_i} P_{h_i}(t)] \\
& + r_i(t)[\alpha_{y_i} y_i^2(t) + \beta_{y_i} y_i(t)] - r_i(t)q_i(t)]dt
\end{aligned}
\tag{6.6.10}
$$

As for the augmented cost functional $J_o(.)$ of (6.3.17), the only changes are in $J_{o_p}(.)$ of (6.3.19) and $J_{o_W}(.)$ of (6.3.30), these will be given by:

$$
\begin{aligned}
J_{o_p}(.) = \int_0^{T_f} [& \sum_{i=N_h+1}^{N_g} [\beta_i + \ell_i'(t) - \ell_i(t) - \lambda_{p_i}(t)]P_i(t) \\
& + \sum_{i=N_h+1}^{N_g} (M_i(t) + \gamma_i)P_i^2(t) \\
& + \sum_{i=1}^{N_h} [\ell_i'(t) - \ell_i(t) - \lambda_{p_i}(t) + m_i(t)\delta_{g_i}] \\
& P_i(t) + \sum_{i=1}^{N_h} [M_i(t) + m_i(t)\gamma_{g_i}]P_i^2(t) \\
& + \sum_{i=1}^{N_h} n_i(t)P_i(t)G_i(t) + \sum_{i=1}^{N_h} m_i(t)\alpha_{g_i} G_i^2(t)
\end{aligned}
$$

$$+ \sum_{i=1}^{N_h} m_i(t)\beta_{g_i} G_i(t)]dt \qquad (6.6.11)$$

$$J_{o_W}(.) = \int_0^{T_f} [\sum_{i=1}^{N_h} [n_i(t)y_{T_{io}} - r_i(t)]q_i(t)$$

$$+ \sum_{i=1}^{N_h} n_i(t)\beta_{T_i} q_i^2(t) - \sum_{i=1}^{N_h} n_i(t)q_i(t)y_i(t)$$

$$+ \sum_{i=1}^{N_h} \dot{r}_i(t)[\alpha_{y_i} y_i^2(t) + \beta_{y_i} y_i(t)]]dt \qquad (6.6.12)$$

The control vector defined in (6.3.31) is modified as follows:

$$\underline{P}_h(t) = \text{col.}[P_1(t), G_1(t), \ldots, P_{N_h}(t), G_{N_h}(t)] \qquad (6.6.13)$$

$$\underline{W}_i(t) = \text{col.}[q_i(t), y_i(t)] \qquad (6.6.14)$$

The last two equations correspond to (6.3.33) and (6.3.40) respectively. The control vector is a $2[n + 1.5N_h + N_g]\text{x}1$ column vector function in this case.

The auxilliary vector $\underline{L}(t)$ of (6.3.32) is modified to:

$$\underline{L}_{P_h}(t) = \text{col.}[\underline{L}_{P_1}(t), \ldots, \underline{L}_{P_{N_h}}(t)] \qquad (6.6.15)$$

$$\underline{L}_{P_i}(t) = \text{col.}[(m_i(t)\delta_{g_i}(t) + \ell'_i(t) - \ell_i(t) - \lambda_{p_i}(t)),$$

$$(m_i(t)\beta_{g_i})] \qquad i = 1, \ldots, N_h \qquad (6.6.16)$$

$$\underline{L}_{W_i}(t) = \text{col.}[(n_i(t)y_{T_{io}} - r_i(t)), \dot{r}_i(t)\beta_{y_i}]$$

$$i = 1, \ldots, N_h \qquad (6.6.17)$$

Finally the matrix $\underline{B}(t)$ of (6.3.47) is modified to:

$$\underline{B}_{P_n}(t) = \mathrm{diag}(\underline{B}_{P_i}(t)) \qquad i = 1,\ldots,N_h \tag{6.6.18}$$

$$\underline{B}_{P_i}(t) = \begin{bmatrix} [M_i(t) + m_i(t)\gamma_{g_i}] & \frac{n_i(t)}{2} \\ \frac{n_i(t)}{2} & m_i(t)\alpha_{g_i} \end{bmatrix}$$

$$i = 1,\ldots,N_h \tag{6.6.19}$$

$$\underline{B}_{W_i}(t) = \begin{bmatrix} \beta_{T_i} n_i(t) & -\frac{n_i(t)}{2} \\ -\frac{n_i(t)}{2} & \alpha_{y_i} \dot{r}_i(t) \end{bmatrix}$$

$$i = 1,\ldots,N_h \tag{6.6.20}$$

These are the changes as far as the formulation is concerned.

The modification in deriving the optimal solution as obtained in Section (6.4) follows easily. First in obtaining $T^*\xi$ of (6.4.4) the following modification is needed to (6.4.14) using (6.6.20)

$$\underline{T}_{W_i} = \left[\frac{\xi_i \alpha_{y_i} \dot{r}_i(t)}{\Delta_{W_i}(t)}, \frac{\xi_i n_i(t)}{2\Delta_{W_i}(t)}\right] \tag{6.6.21}$$

Here
$$\Delta_{W_i}(t) = n_i(t)[\beta_{T_i} \alpha_{y_i} \dot{r}_i(t) - \frac{n_i(t)}{4}] \tag{6.6.22}$$

is the determinant of the matrix in (6.6.20). Second, the operator J is obtained as:

[illegible]

$$J[\underline{\xi}] = \mathrm{col}[(\xi_i \alpha_{y_i} \int_0^{T_f} \frac{\dot{r}_i(t)}{\Delta_{W_i}(t)} dt)] \qquad i = 1,\ldots,N_h \tag{6.6.23}$$

Thus $\underline{\Lambda}$ of (6.4.18) becomes:

$$\underline{\Lambda} = \mathrm{diag}[\alpha_{y_i} \int_0^{T_f} \frac{\dot{r}_i(t)}{\Delta_{W_i}(t)} dt] \qquad i = 1,\ldots,N_h \tag{6.6.24}$$

Finally, $\underline{T^{\dagger}\xi}$ is modified only in:

$$\underline{T^{\dagger}\xi}|_{\underline{W}_i} = \mathrm{col.}[\frac{\xi_i \dot{r}_i(t)}{\Delta_{W_i}(t) \int_0^{T_f} \frac{\dot{r}_i(t)}{\Delta_{W_i}(t)} dt}, \frac{\xi_i n_i(t)}{2\Delta_{W_i}(t) \int_0^{T_f} \frac{\dot{r}_i(t)}{\Delta_{W_i}(t)} dt}]$$

$$i = 1,\ldots,N_h \tag{6.6.25}$$

The expression for $\underline{V}(t)$ as given by (6.4.25) will not be changed. Only the expressions for its components $\underline{V}_{P_h}(t)$ and $\underline{V}_{W_i}(t)$ will be modified. Thus (6.4.34) is modified to:

$$\underline{V}_{P_h}(t) = \mathrm{col.}[V_{p_1}(t), V_{G_1}(t), \ldots, V_{p_{N_h}}, V_{G_{N_h}}(t)] \tag{6.6.26}$$

with:

$$V_{p_i}(t) = \{[m_i(t)\delta_{g_i} + \ell_i'(t) - \ell_i(t) - \lambda_{p_i}(t)]m_i(t)\alpha_{g_i} - 0.5m_i(t)n_i(t)\beta_{g_i}\}/\{\Delta_{W_{P_i}}(t)\} \tag{6.6.27}$$

$$V_{G_i}(t) = \{[M_i(t) + m_i(t)\gamma_{g_i}]m_i(t)\beta_{g_i} - 0.5n_i(t) [m_i(t)\delta_{g_i} + \ell_i'(t) - \ell_i(t) - \lambda_{p_i}(t)]\}/\{\Delta_{W_{P_i}}(t)\} \tag{6.6.28}$$

$$\Delta_{W_{p_i}}(t) = m_i(t)\alpha_{g_i}[M_i(t) + \gamma_{g_i} m_i(t)] - (n_i^2(t)/4) \tag{6.6.29}$$

Also (6.4.46), (6.4.47) and (6.4.48) are modified to:

$$\underline{V}_{W_i}(t) = \text{col.}[V_{W_{q_i}}(t), V_{W_{y_i}}(t)] \tag{6.6.30}$$

$$V_{W_{q_i}}(t) = \{[n_i(t)y_{T_{io}} - r_i(t)]\alpha_{y_i}\dot{r}_i(t) + 0.5n_i(t)\dot{r}_i(t) \beta_{y_i}\}/\{\Delta_{W_i}(t)\} \tag{6.6.31}$$

$$V_{W_{y_i}}(t) = \{0.5n_i(t)[n_i(t)y_{T_{io}} - r_i(t)] + \beta_{T_i}\beta_{y_i} n_i(t)\dot{r}_i(t)\}/\{\Delta_{W_i}(t)\} \tag{6.6.32}$$

The optimal solution as given by (6.4.53) through (6.4.56) is unchanged. However, component-wise only (6.4.57) through (6.4.59) change to:

$$q_{\xi_i}(t) = [-V_{W_{q_i}}(t)/2] + \{[b_i + \int_0^{T_f} [V_{W_{q_i}}(t)/2]dt]\dot{r}_i(t)/ [\Delta_{W_i}(t)\int_0^{T_f} \frac{\dot{r}_i(t)}{\Delta_{W_i}(t)} dt]\} \tag{6.6.33}$$

$$y_{\xi_i}(t) = [-V_{W_{y_i}}(t)/2] + \{[b_i + \int_0^{T_f} [V_{W_{q_i}}(t)/2]dt]n_i(t)/ [2\Delta_{W_i}(t)\int_0^{T_f} \frac{\dot{r}_i(t)}{\Delta_{W_i}(t)} dt]\} \tag{6.6.34}$$

$$P_{h_i}(t) = -V_{p_i}(t)/2 \tag{6.6.35}$$

$$G_{i_\xi}(t) = -V_{G_i}(t)/2 \qquad (6.6.36)$$

Here the variables $q_{\xi_i}(t)$, $y_{\xi_i}(t)$, $P_{h_{i_\xi}}(t)$ and $G_{i_\xi}(t)$ are to satisfy the equality constraints (6.6.5), (6.6.7) and (6.6.8).

CHAPTER VII

CONCLUDING REMARKS

7.1 Conclusions

In this thesis a functional analytic optimization technique is applied to problems of economy scheduling of hydro-thermal electric power systems. Here, the minimum norm formulation is employed to find the optimum generation schedules. This investigation shows how the powerful minimum norm formulation can be applied to complex problems of high dimension.

In Chapters 3 and 4, some simplified economy scheduling problems are considered. The problems are posed and solved using the minimum norm formulation. These problems were investigated earlier using other optimization techniques. However, the solution obtained here is guaranteed to be the unique optimal solution. Moreover, limitations on the unknown functions obtained through this particular formulation facilitates the practical implementation of the optimal solution. A further simplification is the elimination of the multipliers associated with constraints that are linear in the control vector. The solutions obtained here are easily shown to agree with the previously obtained solutions using other methods. This provides a firm ground for investigating more complex problems.

The problems posed and solved in Chapters 5 and 6 represent one main contribution of this investigation. So far as the author of this

thesis knows, there has been little or no work in the following areas: First, the time delay of flow between hydro-plants on the same stream is included in the formulations of Chapter 5. Also, the tail-race elevation effect on the operating hydraulic head is considered here. Second, the formulation in Chapter 6 is in terms of the exact model of the electric network. The reliability objective and the practical limitations on the network variables are also considered. The formulation concerns itself with a hydro-thermal system with variable head hydro-plants. Third, the effects of efficiency variations were incorporated and a trapezoidal reservoir is considered. These formulations in addition enjoy the advantages cited before.

An important aspect of the problems considered in this thesis is the computational schemes adopted. Due to the nonlinearity of the resulting equations, the method is iterative in nature. Here employing the modified contraction mapping principles is very useful. Furthermore, the transformation of the differential equations into operator equations in Chapter 5 guarantees satisfaction of the boundary conditions at each iteration step.

7.2 Suggestions for Further Research

The minimum norm formulation employed in this investigation has demonstrated the capability of solving complex power system scheduling problems. Further research with the same technique would be desirable in order to explore the possibility of solving more complex problems. For example, it may be possible to solve common-flow problems in the case where the time delay of water flow is a function of the rate of water

discharge. The justification of the constant time delay lies in the fact that these delays are of the order of a fraction of the optimization interval considered. Also efforts in the field of defining an overall reliability functional are highly desirable.

The optimization method used in this research also holds promise for related problem areas of the electric power industry. Similar system optimization problems arise in the design of new power transmission systems and in determining favourable locations for building new generation stations. In addition, the system design and optimization of new transmission line parameters as independent variables in the optimization problems are problems that may be handled using this method.

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APPENDIX A

CONVERGENCE CONDITIONS FOR THE ALGORITHM (5.2.227)

Consider the operator equation (5.2.226) of Chapter 5. Then using (5.2.217) through (5.2.220) this is rewritten as:

$$T[\underline{Y}(t)] = H^{V_{MN}}(t)\underline{C} + \int_0^{T_f} G^{V_{MN}}(t,s)[\underline{F}(y(s),s) - \underline{V}(s)\underline{Y}(s)]ds \tag{A.1}$$

where

$$\underline{H}^{V_{MN}}(t)\underline{C} = [\frac{b_2 t}{T_f}, \frac{b_2}{T_f}, \frac{b_3 t}{T_f}, \frac{b_3}{T_f}]^T \tag{A.2}$$

$$\underline{V}(s) = \mathrm{diag}[\underline{V}_1, \underline{V}_1] \tag{A.3}$$

$$\underline{V}_1(s) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\underline{F}(\underline{y}(s),s) = [y_2(s), -f_2(y(s)), y_4(s), -f_3(y(s))]^T \tag{A.4}$$

$$\underline{G}^{V_{MN}}(t,s) = \mathrm{diag}[\underline{G}_a^{V_{MN}}(t,s), \underline{G}_a^{V_{MN}}(t,s)] \tag{A.5}$$

$$\underline{G}_a^{V_{MN}}(t,s) = \begin{bmatrix} (1 - \frac{t}{T_f}) & (-s)(1 - \frac{t}{T_f}) \\ -\frac{1}{T_f} & \frac{s}{T_f} \end{bmatrix} \tag{A.5a}$$

$$G_a^{V_{MN}}(t,s) = \begin{bmatrix} -\frac{t}{T_f} & -\frac{t}{T_f}(T_f - s) \\ -\frac{1}{T_f} & -\frac{1}{T_f}[T_f - s] \end{bmatrix} \quad t < s \leq T_f \tag{A.5b}$$

Under certain conditions [56], the operator in (A.1) is differentiable with derivative given by:

$$T'_{\gamma}[v(t)] = \int_{0}^{T_f} G^{V_{MN}}(t,s)[\frac{\partial F}{\partial \underline{y}}(y,s) - \underline{V}(s)]\underline{v}(s)ds \tag{A.6}$$

An expression for $[\frac{\partial F}{\partial \underline{y}}(y,s) - \underline{V}(s)]$ can be obtained from (A.3) and (A.4). This is given by:

$$[\frac{\partial F}{\partial \underline{y}}(\underline{y}(s),s) - \underline{V}(s)] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{\partial f_2}{\partial y_1} & -\frac{\partial f_2}{\partial y_2} & -\frac{\partial f_2}{\partial y_3} & -\frac{\partial f_2}{\partial y_4} \\ 0 & 0 & 0 & 0 \\ -\frac{\partial f_3}{\partial y_1} & -\frac{\partial f_3}{\partial y_2} & -\frac{\partial f_3}{\partial y_3} & -\frac{\partial f_3}{\partial y_4} \end{bmatrix} \tag{A.7}$$

Thus

$$\underline{G}^{V_{MN}}(t,s)[\frac{\partial F}{\partial \underline{y}}(\underline{y}(s),s) - \underline{V}(s)]\underline{v}(s) = \begin{bmatrix} -\sum_{j=1}^{4} g_{12}(t,s)\ \frac{\partial f_2}{\partial y_j}\ v_j(s) \\ -\sum_{j=1}^{4} g_{22}(t,s)\ \frac{\partial f_2}{\partial y_j}\ v_j(s) \\ -\sum_{j=1}^{4} g_{12}(t,s)\ \frac{\partial f_3}{\partial y_j}\ v_j(s) \\ -\sum_{j=1}^{4} g_{22}(t,s)\ \frac{\partial f_3}{\partial y_j}\ v_j(s) \end{bmatrix} \tag{A.8}$$

Here

$$G_2^{V_{MN}}(t,s) = \begin{bmatrix} g_{11}(t,s) & g_{12}(t,s) \\ g_{21}(t,s) & g_{22}(t,s) \end{bmatrix} \tag{A.9}$$

$$\underline{v}(s) = \text{col.}[v_1(s), v_2(s), v_3(s), v_4(s)] \tag{A.10}$$

Thus (A.6) reduces to:

$$\underline{T}'_Y[v(t)] = \text{col.}[t'_{v_1}, t'_{v_2}, t'_{v_3}, t'_{v_4}] \tag{A.11}$$

$$t'_{v_1} = \int_0^{T_f} - \sum_{j=1}^{4} g_{12}(t,s) \frac{\partial f_2}{\partial y_j} v_j(s)ds \tag{A.12}$$

$$t'_{v_2} = \int_0^{T_f} - \sum_{j=1}^{4} g_{22}(t,s) \frac{\partial f_2}{\partial y_j} v_j(s)ds \tag{A.13}$$

$$t'_{v_3} = \int_0^{T_f} - \sum_{j=1}^{4} g_{12}(t,s) \frac{\partial f_3}{\partial y_j} v_j(s)ds \tag{A.14}$$

$$t'_{v_4} = \int_0^{T_f} - \sum_{j=1}^{4} g_{22}(t,s) \frac{\partial f_3}{\partial y_j} v_j(s)ds \tag{A.15}$$

The modified contraction mapping defined in (5.2.227) is

$$\underline{Y} = \underline{P}\ \underline{Y} = [I-\underline{U}]^{-1}[T[Y] - U(Y)] \tag{A.16}$$

This has a derivative given by:

$$P'_Y[\underline{v}(t)] = [\underline{I}-\underline{U}]^{-1}[T'_Y[v(t)] - \underline{U}\ v(t)] \tag{A.17}$$

In the case when $\underline{U}$ is given by:

$$\underline{U} = \text{diag}[\mu_1, \mu_2, \mu_3, \mu_4] \tag{A.18}$$

Then

$$P'_Y[\underline{v}(t)] = \text{col.}[p'_1, p'_2, p'_3, p'_4] \tag{A.19}$$

$$p'_i(t) = \frac{t'_{v_i} - \mu_i v_i}{1 - \mu_i} \tag{A.20}$$

Now the norm of $P'_Y[\underline{v}(t)]$ is given by

$$||P'_y|| = \underset{||v|| \leq 1}{\text{Sup}}\{||P'_Y[v(t)]||\}$$

or

$$||P_y'|| = \underset{||v||\leq 1}{\mathrm{Sup}} \{\underset{i}{\mathrm{Sup}} \{\underset{t}{\mathrm{Sup}} |p_i'(t)|\}\} \tag{A.21}$$

The convergence conditions for the algorithm (A.16) are

$$||\underline{P}(Y_o) - Y_o|| = \underset{i}{\mathrm{Sup}} \ \underset{t}{\mathrm{Sup}}\{|P(y_o)_i - y_{o,i}(t)|\} \leq \eta, \ \eta \geq 0 \tag{A.22}$$

$$\mathrm{Sup}\{||P_y'||\} \leq \alpha < 1 \tag{A.23}$$

yes

Consider the components of the derivative operator T_Y' as given by (A.12) through (A.15). These yield:

$$|t_{v_1}'| \leq r_{12}(t)Z_2 \tag{A.24}$$

$$|t_{v_2}'| \leq r_{22}(t)Z_2 \tag{A.25}$$

$$|t_{v_3}'| \leq r_{12}(t)Z_3 \tag{A.26}$$

$$|t_{v_4}'| \leq r_{22}(t)Z_3 \tag{A.27}$$

where

$$r_{12}(t) = \int_0^{T_f} |g_{12}(t,s)|ds \tag{A.28}$$

$$r_{22}(t) = \int_0^{T_f} |g_{22}(t,s)|ds \tag{A.29}$$

$$Z_2 = \underset{||v||\leq 1}{\mathrm{Sup}} \ \underset{j}{\mathrm{Sup}} \ \underset{s}{\mathrm{Sup}} \sum_{j=1}^{4} \frac{\partial f_2}{\partial y_j} v_j(s) \tag{A.30}$$

$$Z_3 = \sup_{\|v\|\leq 1} \ \sup_j \ \sup_s \sum_{j=1}^{4} \frac{\partial f_3}{\partial y_j} v_j(s) \tag{A.31}$$

Evaluation of $r_{12}(t)$ and $r_{22}(t)$ yields:

$$r_{12}(t) = \frac{t}{2}[T_f - t] \tag{A.32}$$

$$r_{22}(t) = \frac{1}{2T_f}[(T_f - t)^2 + t^2] \tag{A.33}$$

Moreover

$$\sup_t r_{12}(t) = T_f^{\,2}/8 \tag{A.34}$$

$$\sup_t r_{22}(t) = T_f/2 \tag{A.35}$$

Thus conservative limits on the $|t'_{v_i}|$ given by (A.24) through (A.27) are as follows:

$$|t'_{v_1}| \leq \frac{T_f^{\,2}}{8} Z_2 \tag{A.36}$$

$$|t'_{v_2}| \leq \frac{T_f}{2} Z_2 \tag{A.37}$$

$$|t'_{v_3}| \leq \frac{T_f^{\,2}}{8} Z_3 \tag{A.38}$$

$$|t'_{v_4}| \leq \frac{T_f}{2} Z_3 \tag{A.39}$$

Using (A.20), the following inequality holds true:

$$|p_i'(t)| \leq [1 - \mu_i]^{-1}\{|t'_{v_i}| + |\mu_i v_i(t)|\} \tag{A.40}$$

or using (A.36) through (A.39):

$$|p_1'(t)| \leq [1 - \mu_1]^{-1}\{\frac{T_f^2}{8} Z_2 + |\mu_1|\} \tag{A.41}$$

$$|p_2'(t)| \leq [1 - \mu_2]^{-1}\{\frac{T_f}{2} Z_2 + |u_2|\} \tag{A.42}$$

$$|p_3'(t)| \leq [1 - \mu_3]^{-1}\{\frac{T_f^2}{8} Z_3 + |\mu_3|\} \tag{A.43}$$

$$|p_4'(t)| \leq [1 - \mu_4]^{-1}\{\frac{T_f}{2} Z_3 + |u_4|\} \tag{A.44}$$

Assuming $u_i = \mu$ $(i = 1,\ldots,4)$, then (A.21) reduces to

$$||P_\gamma'|| \leq \underset{i}{\mathrm{Sup}}[\underset{t}{\mathrm{Sup}}\{|p_1'(t)|\}, \underset{t}{\mathrm{Sup}}|p_3'(t)|] \tag{A.45}$$

or

$$||P_\gamma'|| \leq \underset{2,3}{\mathrm{Sup}}\{k_2, k_3\} \tag{A.46}$$

where

$$k_i = [1 - \mu]^{-1}[\frac{T_f^2}{8} Z_i + |\mu|] \qquad i = 2,3 \tag{A.47}$$

The evaluation of estimates on Z_2 and Z_3 is done using (A.30) and (A.31), which yields:

$$Z_i \leq \underset{s}{\mathrm{Sup}}|\sum_{j=1}^{4} \frac{\partial f_i}{\partial y_j}| \tag{A.48}$$

Now (5.2.216) written using the notation of this section is

$$f_2(\underline{y},s) = [\varepsilon_2 y_1(s) + y_2(s)]\rho_2(\underline{y},s) + G_2(\underline{y},s) \tag{A.49}$$

$$f_3(\underline{y},s) = [\varepsilon_3 y_3(s) + y_4(s)]\rho_3(\underline{y},s) + G_3(\underline{y},s) \tag{A.50}$$

so that

so that

$$\sum_{j=1}^{4} \frac{\partial f_2(\underline{y},s)}{\partial y_j} = \rho_2(\underline{y},s)[1 + \varepsilon_2] + [[\varepsilon_2 y_1(s) + y_2(s) + \frac{A_2(s)}{2C_2}] \sum_{j=1}^{4} \frac{\partial \rho_2(\underline{y},s)}{\partial y_j}] + \sum_{j=1}^{4} \frac{\partial g_2(\underline{y},s)}{\partial y_j} \tag{A.51}$$

Here use is made of (5.2.206) for $G_2(\underline{y},s)$. Similarly one obtains:

$$\sum_{j=1}^{4} \frac{\partial f_3(\underline{y},s)}{\partial y_j} = \rho_3(\underline{y},s)[1 + \varepsilon_3] + [[\varepsilon_3 y_3(s) + y_4(s) + \frac{A_3(s)}{2C_3}] \sum_{j=1}^{4} \frac{\partial \rho_3(\underline{y},s)}{\partial y_j}] + \sum_{j=1}^{4} \frac{\partial g_3(\underline{y},s)}{\partial y_j} \tag{A.52}$$

Rewriting (5.2.176) and (5.2.178) in terms of y_1 and y_2, one has

$$g_2(\underline{y},s) = B_3 n_3(t+\tau_1) y_4(t+\tau_1) \qquad t \in [0, T_f-\tau_1)$$

$$= 0 \qquad t \in (T_f-\tau_1, T_f] \tag{A.53}$$

$$g_3(\underline{y},s) = -\frac{d}{ds}[B_3 n_3(s) \psi_2(s,\tau_1)] \qquad t \in [0,\tau_1)$$

$$= -\frac{d}{ds}[B_3 n_3(s)\{\psi_2(\tau_1,\tau_1) + y_1(t-\tau_1)\}]$$

$$t \in [\tau_1, T_f] \tag{A.54}$$

Thus

$$\sum_{j=1}^{4} \frac{\partial g_2(\underline{y},s)}{\partial y_j} = B_3[a_3(t) + b_3(t) \sum_{j=1}^{4} \frac{\partial n_3(\underline{y},s)}{\partial y_j} \tag{A.55}$$

$$\sum_{j=1}^{4} \frac{\partial g_3(\underline{y},s)}{\partial y_j} = -B_3[C_3(t) + d_3(t) \sum_{j=1}^{4} \frac{\partial n_3(\underline{y},s)}{\partial y_j}] \tag{A.56}$$

with

$$a_3(t) = [\dot{y}_4(t+\tau_1)n_3(t+\tau_1)/\dot{y}_4(t)] \qquad t\ \varepsilon[0,T_f-\tau_1)$$

$$= \ 0 \qquad t\ \varepsilon(T_f-\tau_1,T_f]$$

(A.57)

$$b_3(t) = y_4(t+\tau_1)\dot{n}_3(t+\tau_1)/\dot{n}_3(t) \qquad t\ \varepsilon[0,T_f-\tau_1)$$

$$= \ 0 \qquad t\ \varepsilon(T_f-\tau_1,T_f)$$

(A.58)

$$C_3(t) = [\{\dot{n}_3(t)\dot{y}_1(t-\tau_1)/\dot{y}_1(t)\} + \{n_3(t)\dot{y}_2(t-\tau_1)/\dot{y}_2(t)\}]$$

$$t\ \varepsilon[\tau_1,T_f] \qquad \text{(A.59)}$$

$$d_3(t) = [y_2(t-\tau_1) + \{\ddot{n}_3(t)[\psi_2(\tau_1,\tau_1) + y_1(t-\tau_1)]/\dot{n}_3(t)\}]$$

$$t\ \varepsilon[\tau_1,T_f] \qquad \text{(A.60)}$$

Moreover since

$$\rho_i(\underline{y},s) = \dot{n}_i(s)/n_i(s)$$

then

$$\sum_{j=1}^{4} \frac{\partial\rho_i(\underline{y},s)}{\partial y_j} = [\frac{\ddot{n}_i}{\dot{n}_i} - \rho_i]\,\frac{1}{n_i}\ \sum_{j=1}^{4}\ \frac{\partial n_i}{\partial y_k} \qquad \text{(A.61)}$$

Using (A.55),(A.56) and (A.61) in (A.51) and (A.52) one obtains:

$$\sum_{j=1}^{4} \frac{\partial f_2(\underline{y},s)}{\partial y_j} = \rho_2(\underline{y},s)[1 + \varepsilon_2] + B_3a_3(t)$$

$$+ \{[\varepsilon_2y_1(s) + y_2(s) + \frac{A_2(s)}{2C_2}][\frac{\ddot{n}_2}{\dot{n}_2} - \rho_2]$$

$$[\frac{1}{n_2}] \sum_{j=1}^{4} \frac{\partial n_2}{\partial y_j}\} + B_3 b_3(t) \sum_{j=1}^{4} \frac{\partial n_3}{\partial y_j} \quad \text{(A.62)}$$

$$\sum_{j=1}^{4} \frac{\partial f_3(\underline{y},s)}{\partial y_j} = \rho_3(y,s)[1 + \varepsilon_3] - B_3 C_3(t)$$

$$+ [\frac{1}{n_3} [\frac{\ddot{n}_3}{\dot{n}_3} - \rho_3][\varepsilon_3 y_3(s) + y_4(s) + \frac{A_3(s)}{2C_3}]$$

$$- B_3 d_3(t)] \sum_{j=1}^{4} \frac{\partial n_3}{\partial y_j} \quad \text{(A.63)}$$

From (5.2.197), (5.2.199) and (5.2.200) with $B_{ij} = 0 \quad i \neq j$ one obtains:

$$n_2(t) = \lambda(t)[1 - 2B_{22}P_{h_2}(t)] \quad \text{(A.64)}$$

$$n_3(t) = \lambda(t)[1 - 2B_{33}P_{h_3}(t)] \quad \text{(A.65)}$$

$$P_D(t) + B_{11}[\frac{\lambda(t) - \beta}{2[\gamma + \lambda(t)B_{11}]}]^2 + B_{22}P_{h_t}{}^2(t) + B_{33}P_{h_3}{}^2(t)$$

$$- \frac{\lambda(t) - \beta}{2[\gamma + \lambda(t)B_{11}]} - P_{h_2}(t) - P_{h_3}(t) = 0 \quad \text{(A.66)}$$

Moreover, (5.2.201) and (5.2.202) are rewritten as:

$$- P_{h_2}(t) = A_2(t)y_2(t) + B_2 y_1(t)y_2(t) + C_2 y_2{}^2(t) \quad \text{(A.67)}$$

$$- P_{h_3}(t) = A_3(t)y_4(t) + B_3 y_4(t)[y_1(t-\tau_1) - y_3(t)]C_3 y_4{}^2(t) \quad \text{(A.68)}$$

The evaluation of $\sum_{j=1}^{4} \frac{\partial n_i}{\partial y_j}$ can be easily effected utilizing the above euqlities. Thus one obtains:

$$\sum_{j=1}^{4} \frac{\partial n_i}{\partial y_j} = - x_i(t) \sum_{j=1}^{4} \frac{\partial P_{h_i}}{\partial y_j} \qquad i = 2,3 \tag{A.69}$$

$$x_i(t) = 2[\lambda(t)B_{ii} + (\frac{\partial + \lambda(t)B_{11}}{\partial + \beta B_{11}})^2(1 - 2B_{ii}P_{h_i}(t))^2] \tag{A.70}$$

$$\sum_{j=1}^{4} \frac{\partial P_{h_2}}{\partial y_j} = [\frac{P_{h_2}(t)}{y_2(t)} - (B_2 + C_2)y_2(t)] \tag{A.71}$$

$$\sum_{j=1}^{4} \frac{\partial P_{h_3}}{\partial y_j} = [\frac{P_{h_3}(t)}{y_4(t)} - [B_3[\frac{y_2(t-\tau_1)}{y_2(t)} - 1] + C_3]y_4(t)] \tag{A.72}$$

Thus (A.62) is rewritten as:

$$\begin{aligned}\sum_{j=1}^{4} \frac{\partial f_2}{\partial y_j} &= \rho_2(y,s)[1 +\varepsilon_2] + B_3a_3(t) + \frac{x_2(t)}{2C_2n_2(t)} \\ &\quad [\frac{\ddot{n}_2}{\dot{n}_2} - \rho_2][\sum_{j=1}^{4} \frac{\partial P_{h_2}}{\partial y_j}][\sum_{j=1}^{4} \frac{\partial P_{h_2}}{\partial y_j} + B_2y_2] \\ &\quad - B_3b_3(t) \sum_{j=1}^{4} \frac{\partial P_{h_2}}{\partial y_j}\end{aligned} \tag{A.73}$$

Also (A.63) reduces to:

$$\begin{aligned}\sum_{j=1}^{4} \frac{\partial f_3}{\partial y_j} &= \rho_3(1 + \varepsilon_3) - B_3C_3(t) + x_3(t) \sum_{j=1}^{4} \frac{\partial P_{h_3}}{\partial y_j} \\ &\quad [B_3 + \frac{1}{2C_3n_3(t)} [\frac{\ddot{n}_3}{\dot{n}_3} - \rho_3][\sum_{j=1}^{4} \frac{\partial P_{h_3}}{\partial y_j} + B_3(\frac{y_2(t-\tau_1)}{y_2(t)} - 1)]]\end{aligned} \tag{A.74}$$

Using (A.70), then the limit on $x_i(t)$ is obtained as

$$0 \leq x_i(t) \leq x_{i_M} \tag{A.75}$$

$$x_{i_M} = 2[\lambda_M B_{ii} + (1 - 2B_{ii}P_{h_{i_{Max}}})^2[\frac{\gamma + \lambda_M B_{11}}{\gamma + \beta B_{11}}]^2] \tag{A.76}$$

Also (A.71) and (A.72) yield:

$$0 \leq \sum_{j=1}^{4} \frac{\partial P_{h_i}}{\partial y_j} \leq \delta_{i_M} \tag{A.77}$$

$$\delta_{i_M} = \frac{P_{h_{i_{Max}}}}{y_{k_{Min}}} \qquad k = 2 \text{ for } i = 2 \text{ and } k = 4 \text{ for } i = 3 \tag{A.78}$$

Let

$$\theta_{i_{Max}} = \underset{t}{\mathrm{Max}}[\frac{\ddot{n}_i}{\dot{n}_i} - \rho_i] \tag{A.79}$$

$$\theta_{i_{Min}} = \underset{t}{\mathrm{Min}}[\frac{\ddot{n}_i}{\dot{n}_i} - \rho_i] \tag{A.80}$$

$$|B_3(\frac{y_2(t-\tau_1)}{y_2(t)} - 1)| \leq k_2 \tag{A.81}$$

Thus (A.73) and (A.74)yield

$$|\sum_{j=1}^{4} \frac{\partial f_2}{\partial y_j}| \leq |\rho_2(1 + \varepsilon_2)| + |B_3 \underset{t}{\mathrm{Max}}\, a_3(t) + \frac{x_{i_M}\theta_{2_{Max}}\delta_{2_M}}{2C_2 \underset{t}{\mathrm{Min}}\, n_2(t)}$$

$$[\delta_{i_M} + B_2 y_{2_M}]| \tag{A.82}$$

$$|\sum_{j=1}^{4} \frac{\partial f_3}{\partial y_j}| \leq |\rho_3(1 + \varepsilon_3)| + |B_3 \underset{t}{\mathrm{Max}}\, C_3(t) + x_{3_M}\delta_{3_M}$$

$$[B_3 + \frac{\theta_{3_{Max}}}{2C_3 \mathrm{Min}\, n_3(t)}[\delta_{i_M} + k_2]]| \tag{A.83}$$

Then (A.30) and (A.31) yield:

$$Z_2 = \text{Max}\ \{|\rho_2(1+\varepsilon_2)|, |B_3 \underset{t}{\text{Max}}\ a_3(t) + \frac{x_{2_M}\theta_{2_{Max}}\delta_{2_M}}{2C_2 \underset{t}{\text{Min}}\ n_2(t)}$$

$$[\delta_{2_M} + B_2 y_{2_M}]|\} \tag{A.84}$$

$$Z_3 = \text{Max}\{|\rho_3(1+\varepsilon_3)|, |B_3 \underset{t}{\text{Max}}\ C_3(t) + x_{3_M}\delta_{3_M}$$

$$(B_3 + \frac{\theta_{3_{Max}}}{2C_3 \underset{t}{\text{Min}}\ n_3(t)}[\delta_{3_M} + k_2]|\} \tag{A.85}$$

Equations (A.84) and (A.85) together with (A.46, (A.85) and (A.47) define the convergence condition (A.23).

The convergence condition (A.22) using (A.1) and (A.16) reduces to:

$$\underset{i}{\text{Sup}}\ \underset{t}{\text{Sup}}\ \{|\Delta_i(t)|\} \leq \eta \qquad \eta \geq 0 \tag{A.86}$$

$$\Delta_i(t) = \frac{1}{(1-\mu_i)}[t_i(\underline{y}_o) - y_{io}] \qquad i = 1,2,\ldots,4 \tag{A.87}$$

Here (A.1) is expressed as

$$T[Y] = \text{col.}[t_1(\underline{y}), t_2(\underline{y}), t_3(\underline{y}), t_4(\underline{y})] \tag{A.88}$$

Further (A.87) reduces to:

$$\Delta_1(t) = \frac{1}{(1-\mu_1)T_f}[\int_0^t s[T_f - t]f_2(y_o,s)ds + \int_0^{T_f} t[T_f - s]f_2(y_o,s)ds] \tag{A.89}$$

$$\Delta_2(t) = \frac{1}{(1-\mu_2)T_f}\left[\int_0^t -sf_2(y_o,s)ds + \int_t^{T_f} (T_f-s)f_2(y_o,s)ds\right] \tag{A.90}$$

$$\Delta_3(t) = \frac{1}{(1-\mu_3)T_f}\left[\int_0^t s[T_f-t]f_3(y_o,s)ds + \int_t^{T_f} t[T_f-s]\right.$$

$$\left. f_3(y_o,s)ds\right] \tag{A.91}$$

$$\Delta_4(t) = \frac{1}{(1-\mu_4)T_f}\left[\int_0^t -sf_3(y_o,s)ds + \int_t^{T_f} [T_f-s]f_3(y_o,s)ds\right] \tag{A.92}$$

These in turn yield:

$$\sup_t |\Delta_1(t)| \leq \frac{2T_f^2}{1-\mu_1}[\mathrm{Sup}|f_2(y_o,t)|] \tag{A.93}$$

$$\sup_t |\Delta_2(t)| \leq \frac{2T_f}{1-\mu_2}[\sup_t|f_2(y_o,t)|] \tag{A.94}$$

$$\sup_t |\Delta_3(t)| \leq \frac{2T_f^2}{1-\mu_3}[\sup_t|f_3(y_o,t)|] \tag{A.95}$$

$$\sup_t |\Delta_4(t)| \leq \frac{2T_f}{1-\mu_4}[\sup_t|f_3(y_o,t)|] \tag{A.96}$$

Let

$$\eta_o = \frac{2T_f^2}{1-\mu}\,\mathrm{Max}\,\{\sup_t|f_2(y_o,t)|,\ \sup_t|f_3(y_o,t)|\} \tag{A.97}$$

then

$$\eta_o \leq \eta \qquad\qquad \eta \geq 0 \tag{A.98}$$

The modified contraction mapping sequence (5.2.227) based on the $\underline{y}_o$ given in (5.2.228) through (5.2.231) converges if condition (A.26) is satisfied. Note that condition (A.98) is satisfied for the indicated

value of $\underset{\sim}{y}_0$. If one is interested in defining the region of convergence, then η must be calculated. However, the crucial convergence inequality is (A.46).